A few t-Spanners in the Euclidean plane[1]

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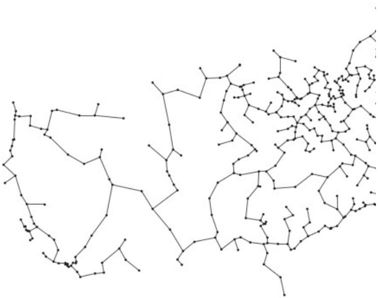
[1] these slides are last updated in 2013; in presenting, blackboard is used

Outline

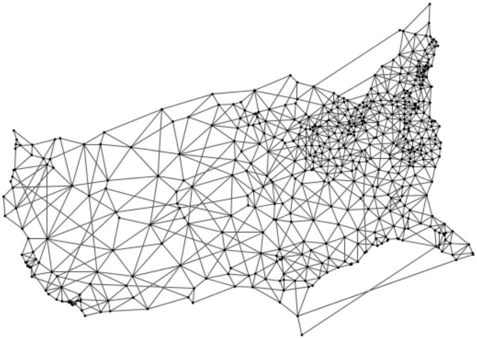
t-Spanner: definition

Given a set S of points in Euclidean plane, network $G(S,E)$ is a *t-spanner* ($t > 1$) of S iff for every $u, v \in V$, $dist_G(u,v) \leq t.dist(u,v)$.

Motivation: designing road networks



10-stretch network for US cities



1.2-stretch network for US cities

Making a network by connecting points given in Euclidean plane. [2]

[2] example figs are from [Narasimhan, Smid '07]

Motivation

Asymptotic improvement for algorithms that rely on m.

Problem

Given a set S of points in Euclidean plane, construct a sparse network $G(S, E)$ that obeys one/many of the following factors:

- low stretch factor ($t = max_{p,q \in S} \frac{\delta_G(p,q)}{\delta(p,q)}$)
- $O(|S|)$ number of edges (*sparse*)
- low weight ($\sum_{e \in E} w_e$)
- minimize the maximum degree (*small size*)
- low diameter (*conciseness*)
- high fault tolerance
- small load factor
- small chromatic number

Problem description

Let S be a set of n points in R^2, and let $t > 1$ be a real number.

- Does there exist a t-spanner for S having at most $O(n)$ edges?
- If so, find the lower and upper bounds in constructing the same.
- Is it possible to construct t-spanners in $O(n \lg n)$ time?
- Weight of such spanner as compared with MST?
- Can we minimize the diameter of the spanner?
- Can we minimize the maximum degree of the spanner?

An application: minimum Steiner tree

Given a set S of points and S' of Steiner points,

wight of Steiner minimum tree of S

$\leq$ weight of minimum spanning tree of S

$\leq$ 2.weight of Steiner minimum tree of S [3]

[3] when points are in $\mathbb{R}^2$, factor 2 got improved to $\frac{2}{\sqrt{3}}$ [Du, Hwang '90]

Spanner degree vs diameter

- Any t-spanner whose degree is bounded by a constant must have a spanner diameter $\Omega(\lg n)$

Another application: t-approximate MST

Let G be a t-spanner of S. Then $wt(MST(G)) \leq t.wt(MST(S))$.

> unioning paths in G corresp. to each edge of $MST(S)$ results in a connected spanning graph G' that is a subgraph of G

Lower bounds

In the algebraic computation tree model, the worst-case lower bound in constructing a spanner of n points stands at $\Omega(n \lg n)$.

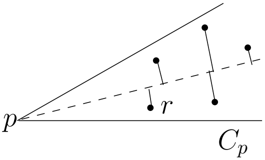
— not proved

Outline

Observation

For each $p \in S$, among all near-parallel edges incident on p in the complete graph, the Θ-graph retains only the shortest one.

Algorithm: preprocessing



input: set S of points in R^2, number of cones κ ($\kappa \geq 9$; so that the cone angle $\theta = \frac{2\pi}{\kappa} \in (0, \frac{\pi}{4})$)

let $\mathcal{C}$ be a set of κ cones partitioning the space around origin

1. introduce each point in S as a vertex in Θ-graph
2. for each point p of S and for each cone C of $\mathcal{C}$, such that the translated cone C_p contains at least one point of $S \backslash \{p\}$, introduce an edge (p, r) into Θ-graph if r is a closest point along the bisector of C to p among all the points in C_p [4]

output: undirected graph $\Theta(S, E)$ with $|E|$ is $O(n\kappa)$

[4] in case of Yao graphs, among all the points in C_p is chosen, closest point to p is chosen

Algorithm: query

input: two query points p and q in S

(1) pick a cone C in $\mathcal{C}$ such that $q \in C_p$

(2) for $r \in C_p$ and pr being an edge of Θ-graph, output r

(3) if $r \neq q$, set $p \leftarrow r$ and go to the stmt (1)

output: a path between p and q

Analysis: stretch factor

for any point q in C_p and closest point r to p along bisector of C_p,

- $|pr| \leq \frac{|pq|}{\cos\theta}$
- $|rq| \leq |pq| - (\cos\theta - \sin\theta)|pr|$

Analysis: stretch factor (cont)

The stretch factor $t = 1/(\cos\theta - \sin\theta)$:

Let $p = p_0, p_1, \ldots, p_m = q$ be the path constructed by the query algorithm.

- $|p_{i+1}q| < |p_i q|$

  implying that each successive point on this path takes us strictly closer to q

- $|p_i p_{i+1}| \leq \frac{1}{\cos\theta - \sin\theta}(|p_i q| - |p_{i+1} q|)$

For each real constant $t > 1$, there exists a sparse t-spanner.

Analysis: complexity

- $O(\kappa n \lg n)$ time (using plane sweep)
- using $O(n\kappa)$ space

Optimizing other parameters

max degree of the spanner

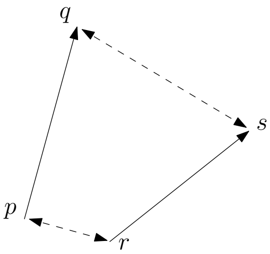
- sink spanner

diameter of the spanner

- skip-list spanner

Outline

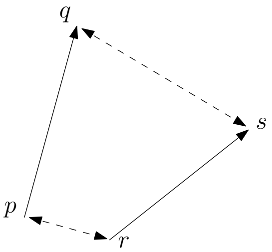
Gap property



Let $w \geq 0$ be a real number, and let E be a set of directed edges in R^d

- E satisfies *w-gap property* whenever for any two distinct edges (p, q) and (r, s) in E, we have $|pr| > w \min(|pq|, |rs|)$
- E satisfies *strong w-gap property* whenever E satisfies w-gap property together with $|qs| > w \min(|pq|, |rs|)$

Gap theorem



Let S be a set of n points in R^d, and let $E \subseteq S \times S$ be a set of directed edges that satisfy the w-gap property.

- if $w \geq 0$, then each point of S is the source of at most one edge of E
- if $w \geq 0$, and E satisfies the strong w-gap property, then each point of S is the sink of at most one edge of E.

Gap theorem (cont)

Let S be a set of n points in R^d, and let $E \subseteq S \times S$ be a set of m directed edges that satisfy the w-gap property. If $w > 0$, then
$wt(E) < (1 + \frac{2}{w}).wt(MST(S)) \lg n$.

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- claim: E contains a subset E' of size $\frac{m}{2}$, such that $wt(E') < (\frac{2}{w})wt(MST(S))$.

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  $wt(E') < (\frac{2}{w})wt(MST(S))$.
  - * number of edges of E according to the order in which their sources are visited by an optimal TSP of S; consider the portion $T_i = (p_{k_{2i-1}}, p_{k_{2i-1}} + 1, \ldots, p_{k_{2i}})$ of TSP(S) between the sources of two successive edges e_{2i-1} and e_{2i}, where $1 \leq i \leq m/2$

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  - $|p_{k_{2i-1}} p_{k_{2i}}| \leq wt(T_i)$ and
    $|p_{k_{2i-1}} p_{k_{2i}}| > w \min(|e_{2i-1}|, |e_{2i}|) \Rightarrow \min(|e_{2i-1}|, |e_{2i}|) < \frac{1}{w} \cdot wt(T_i)$

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  - * $\sum_{i=1}^{m/2} \min(|e_{2i-1}, e_{2i}) \leq \frac{1}{w} \cdot wt(TSP(S))$

Gap theorem (cont)

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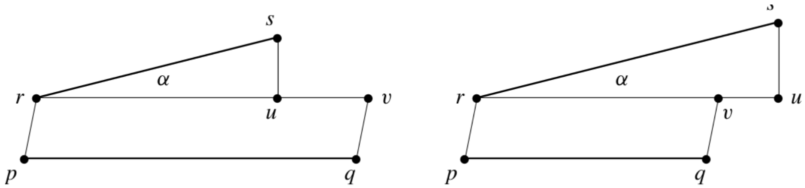
  - $\sum_{i=1}^{m/2} \min(|e_{2i-1}, e_{2i}) \leq \frac{1}{w} \cdot wt(TSP(S))$
- by induction on m

Observation



Let t, θ, and w be real numbers, such that $0 < \theta < \pi/4$, $0 \leq w < (\cos\theta - \sin\theta)/2$, and $t \geq 1/(\cos\theta - \sin\theta - 2w)$. Let p, q, r, and s be points in R^d, such that

- $p \neq q, r \neq s$,
- $angle(pq, rs) \leq \theta$, (r, s) is almost parallel to (p, q)
- $|rs| \leq |pq|/\cos\theta$, $|rs|$ is not much larger than $|pq|$
- $|pr| \leq w|rs|$. r is close to p

Then $|pr| < |pq|$, $|sq| < |pq|$, and $t|pr| + |rs| + t|sq| \leq t|pq|$. — not proved

Another observation

Let θ, w, and t be real numbers such that $0 < \theta < \pi/4$, $0 \leq w < (\cos\theta - \sin\theta)/2$, and $t \geq 1/(\cos\theta - \sin\theta - 2w)$. Let S be a set of n points in the plane, and let $G(S, E)$ be a directed graph, such that the following holds: for any two distinct points p and q of S, there is an edge $(r, s) \in E$, such that

- $angle(pq, rs) \leq \theta$
- $|rs| \leq |pq| / \cos\theta$
- $|pr| \leq w|rs|$ or $|qs| \leq w|rs|$.

Then, the graph G is a t-spanner for S.

The gap-greedy algorithm[5]

Consider all ordered pairs of distinct points in nondecreasing order of their distances. An edge (p, q) is added iff including (p, q) into the current edge set E does not make the new set to violate the w-strong gap property.

[5] from [Arya, Smid '97]

Analysis

when $0 < \theta < \pi/4$ and $0 \leq w \leq (\cos\theta - \sin\theta)/2$,

- stretch factor $1/(\cos\theta - \sin\theta - 2w)$

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- weight $\leq \lceil 2\pi/\theta \rceil (1 + 2/w) wt(MST(S)) \lg n$

Analysis

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- maximum degree $\leq 2\lceil 2\pi/\theta \rceil$
- weight $\leq \lceil 2\pi/\theta \rceil (1 + 2/w) wt(MST(S)) \lg n$
- construction time $O(n^3)$

Analysis: optimizing parameters to minimize the weight

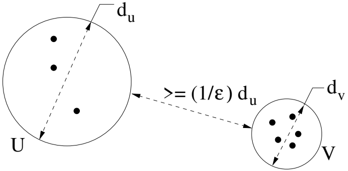
when $\theta = (t-1)/2$ and $w = (t-1)/4$,

- stretch factor t
- maximum degree is $O(1/(t-1))$
- weight is $O((1/(t-1)^2) \cdot wt(MST(S)) \lg n)$
- construction time $O(n^3)$ [6]

[6] a modified implementation yields $O(n(\lg n)^2)$ time.

Outline

WSPD: Definition



sets that are $(1/\epsilon)$-separated.

For a point set P, a *well-separated pair decomposition (WSPD)* of P with parameter ϵ is a set of pairs $W = \{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ such that

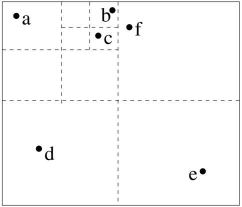
(1) $\forall_i A_i, B_i \subset P$

(2) $\forall_i A_i \cap B_i = \phi$

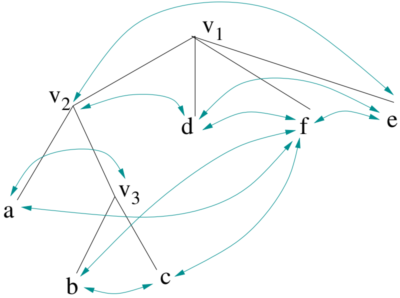
(3) $\bigcup_i A_i \bigotimes B_i = P \bigotimes P = \{\{x, y\} | x \in P, y \in P, x \neq y\}$

(4) $\forall_i A_i, B_i$ are $(1/\epsilon)$-separated.

Using quadtree to compute a WSPD



(a) point set



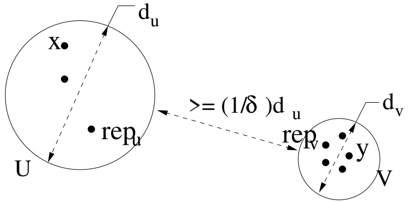
(b) WSPD with $\epsilon = \frac{1}{2}$

For the sake of efficiency, we retrieve a WSPD from compressed quadtree.

size of WSPD: $O((\frac{1}{\epsilon})^d n)$; construction time: $O(n \lg n + \frac{n}{\epsilon^d})$

Spanner construction using WSPD

choose a representative $rep_R \in R$ for every set R in $(1/\delta)$-WSPD; for every $\{U, V\} \in (1/\delta)$-WSPD, add an edge between $rep_U \in U$ and $rep_V \in V$ resulting in $(1+\epsilon)$-spanner G of S.



induction on the increasing length of pairwise distances' of points in P
$d_G(x, y) \leq (1+\epsilon) dist(x, y)$

- $dist(rep_U, rep_V) \leq (1+2\delta) dist(x, y)$
- $max(dist(rep_U, x), dist(rep_V, y)) \leq \delta dist(rep_U, rep_V)$
- further, to apply induction hypotheses, choose a δ such that $max(dist(rep_U, x), dist(rep_V, y)) < dist(x, y)$

Analysis

- $O(n)$ size (in practice, size grows with n much faster than Θ-graph based or greedy algorithms)

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Analysis

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- weight is $O((\lg n)wt(MST))$
- degree is $O(n)$
- construction time is $O(n \lg n)$

Outline

The path-greedy algorithm[7]

Construct spanner G while onsidering pairs of points in nondecreasing order of distances: add an edge e between the considered pair (u, v) only if $d_G(u, v) > t.d(u, v)$.

[7] from [Das, Heffernan, Narasimhan '93]

Analysis

- t-spanner
- $O(n^2(m + n \lg n))$ time (improved algorithms that compute only $O(n)$ SSSPs do exist)

Analysis (cont)

- $O(n)$ size
- constant degree
- weight is $O((\lg n)w(MST))$

— not proved: analysis is a bit involved

has many good characteristics but the computation time is high

Outline

Comparison of t-spanners

- Greedy $\leftarrow$ excellent features but the construction time is the bottleneck
- Θ-graph based
- WSPD based

Current research

- specialized: plane, single-source, pairwise
- amid obstacles
- Steiner
- in R^3
- expected analysis
- dynamic spanners
- kinetic spanners
- energy-efficient
- multicriteria

$\dots$

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Thanks!