Lipton & Tarjan's Planar Separator Theorem[1]

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[1] Prepared in '10; References: The Design and Analysis of Algorithms by D. C. Kozen.

Let $G(V, E)$ be an undirected planar graph (with $|V| \geq 3$). There exists a partition of V into disjoint sets A, B and S such that:

- $|A|, |B| \leq \frac{2n}{3}$
- $|S| \leq 4\sqrt{|V|}$
- $(A \times B) \cap E = \phi$
- Moreover, such a partition can be found in linear time.

Outline

Maximum cardinality matching in planar graphs

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- if G contains no augmenting path with end node v, then M is a maximum matching in G
- otherwise, for an augmenting path P, $M \oplus P$ is a maximum matching in G.

Maximum cardinality matching in planar graphs (cont)

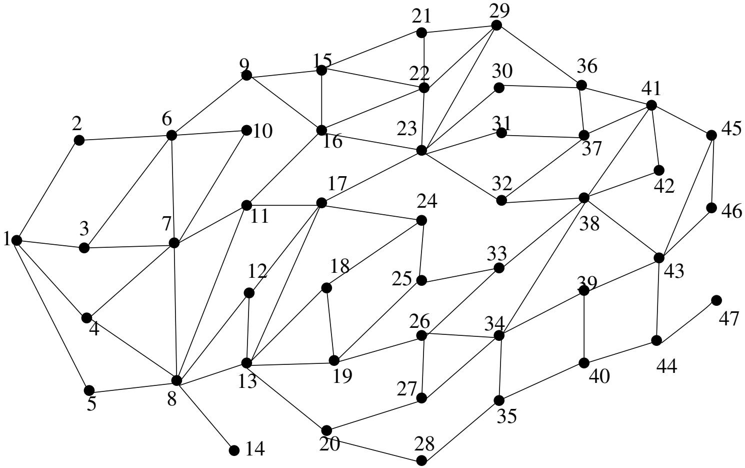
Recursively do the following: divide G using planar separator theorem; conquer the separated pieces; for each vertex in the separator, apply the above theorem to combine.

Leads to $T(n) = 2T(\frac{2}{3}n) + O(n^{3/2})$; solving which yields $O(n^{1.709})$[2]

[2] more precise analysis of this algorithm leads to $O(n^{1.5})$

Outline

Embed G in plane

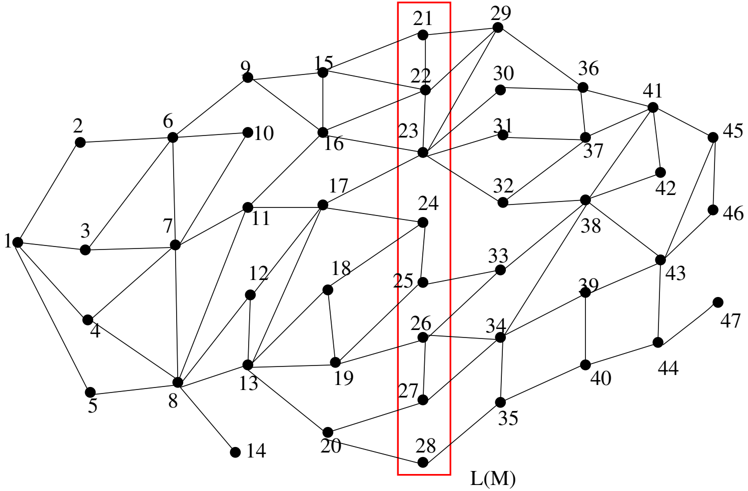


[3]

- In linear time using Hopcroft-Tarjan's algorithm.

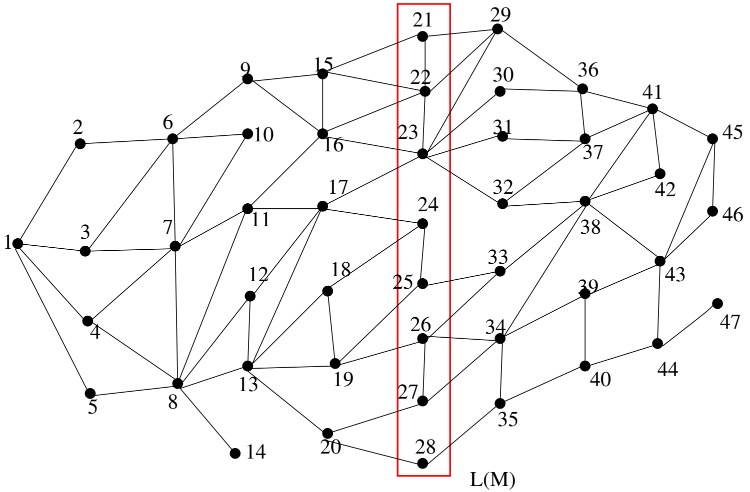
[3] figures in this lecture are from Kozen's text

Find median level using BFS



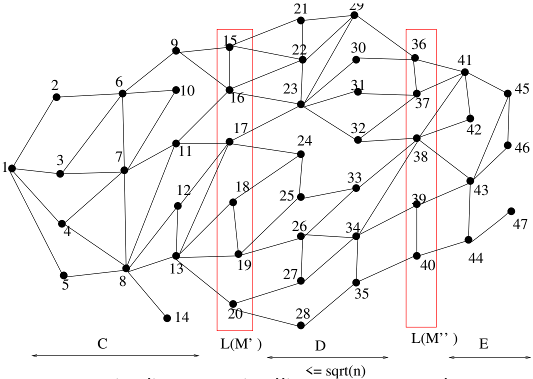
- Find the median level, say M, in which $\frac{n}{2}$th element resides in breadth-first ordering.

Find median level using BFS



- Find the median level, say M, in which $\frac{n}{2}$th element resides in breadth-first ordering.
- Can $L(M)$ be a *valid* separator S?

Find two closest levels $L(M')$ and $L(M'')$ to $L(M)$



- There exists levels $L(M')$ and $L(M'')$ such that $M' \leq M$ and $M'' > M$ and $|L(M')| \leq \sqrt{n}$, $|L(M'')| \leq \sqrt{n}$, and $L(M'') - L(M') \leq \sqrt{n}$.

  introduce a dummy level with zero nodes as the last layer to guarantee this

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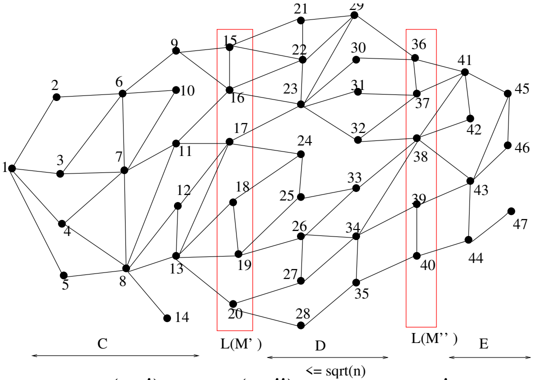

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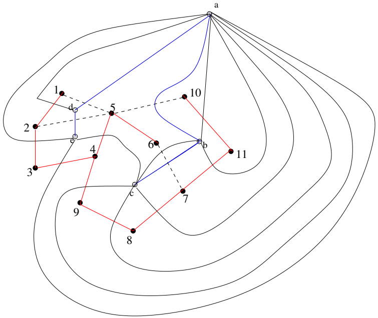
  introduce a dummy level with zero nodes as the last layer to guarantee this

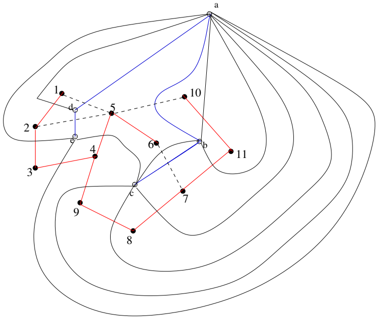
- Can $L(M') \cup L(M'')$ be a *valid* separator S?
  - * yes it is provided $|D| \leq \frac{2}{3}n$
  - * otherwise, since $|C| + |E| \leq \frac{n}{3}$, we find a $\frac{1}{3}$-$\frac{2}{3}$ separator X-Y of D with $\leq 2\sqrt{n}$ vertices and combine this with $L(M')$ and $L(M'')$ to get a separator S of interest; and combine X, Y, C, E appropriately

A property of plane graph and its dual



Let $G(V,E)$ be a connected plane triangulated graph, and let $G^*(V^*,E)$ be its dual. Also, let $E' \subseteq E$.

A property of plane graph and its dual



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- The subgraph (V,E') of G has a cycle if and only if the subgraph $(V^*,E-E')$ of G^* is disconnected.

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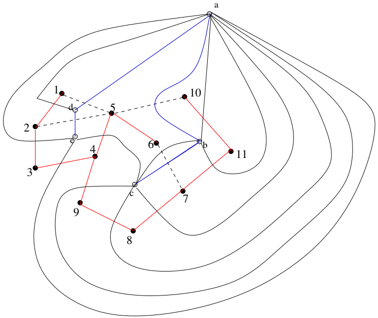

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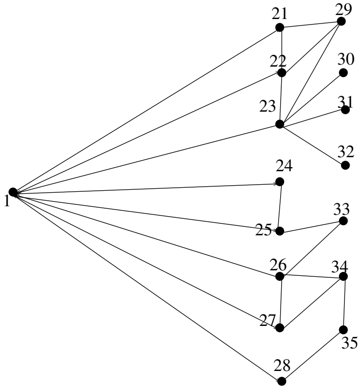
- The subgraph (V, E') of G has a cycle if and only if the subgraph $(V^*, E - E')$ of G^* is disconnected.
- (V, E') is a spanning tree in G if and only if $(V^*, E - E')$ is a spanning tree in G^*.

  For a set E' of edges of a spanning tree T of G, the edges in $E - E'$ are *fronds*; all the fronds together define a spanning tree $T^*(V^*, E - E')$ in G^*.

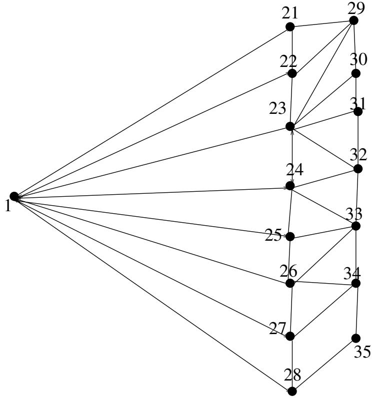
Role of graph induced by nodes in D



source vertex connected to vertices in level $M' + 1$

- Now the objective is to compute a $\frac{1}{3}$-$\frac{2}{3}$ separator for D in the above figure.

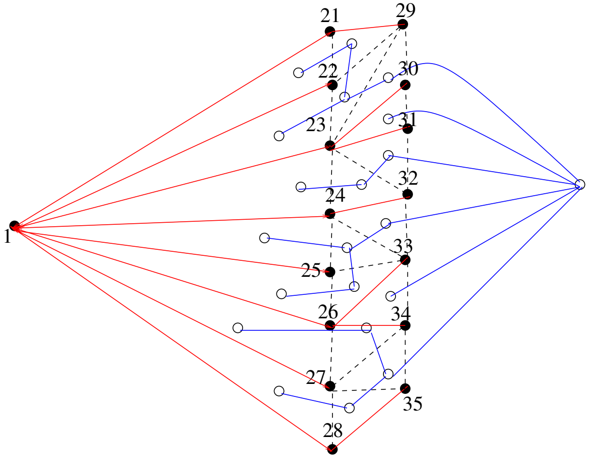
Triangulating graph induced by nodes in $D \cup \{1\}$



- Obtain a triangulation *TR* to utilize the property mentioned above.
- Compute a spanning tree T^* of D^* from a spanning tree T of *TR*.

  Make an arbitrary node of T^* as the root of T^*; and, orient all the edges of T^* away from the root.

Construct spanning tree T^* in D^* from a spanning tree T of TR



- For any frond $f(u,v)$, unique path from u to v in T is a separator for TR, thought it may not necessarily a *valid* separator.

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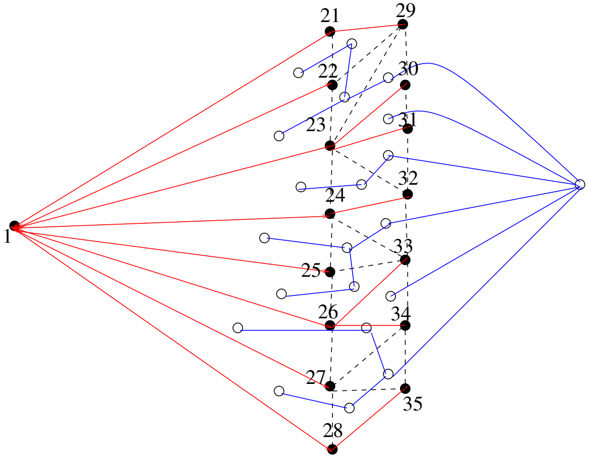
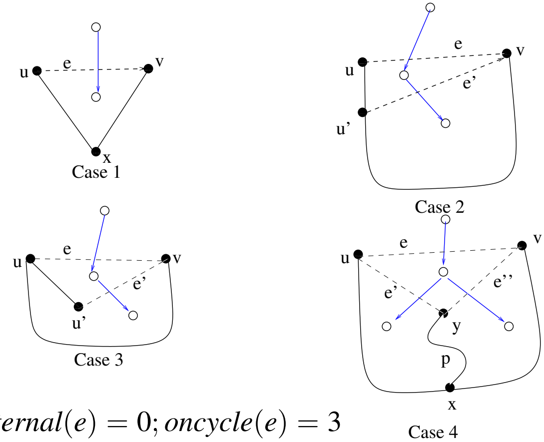


- For any frond $f(u, v)$, unique path from u to v in T is a separator for TR, thought it may not necessarily a *valid* separator.
- Since the diameter of T is $\leq 2\sqrt{n}$, the cardinality of separator that correspond to a frond is upper bounded as well.

Do DFS on T^* to find separator for D



- case 1: $internal(e) = 0; oncycle(e) = 3$
- case 2: $internal(e) = internal(e'); oncycle(e) = oncycle(e') + 1$
- case 3: $internal(e) = internal(e') + 1; oncycle(e) = oncycle(e') - 1$
- case 4: $internal(e) = internal(e') + internal(e'') + |p| - 1; oncycle(e) = oncycle(e') + oncycle(e'') - 2|p| + 1$

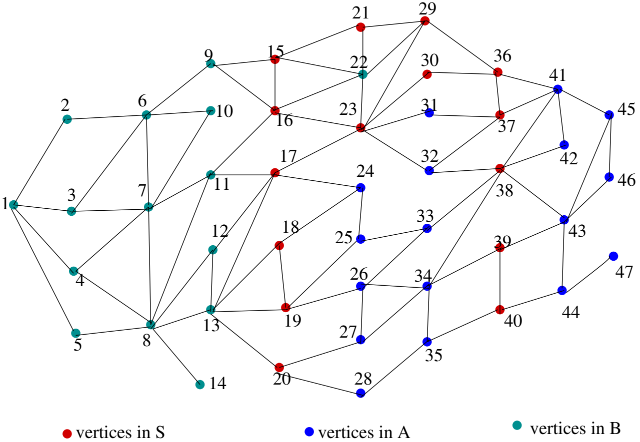
and, maintain nodes on each cycle in each case (in linear time)

$\exists$ a frond that correspond to a $\frac{1}{3}$-$\frac{2}{3}$ separator for D

For the first frond e encountered on the way out from the leaves of T^* to the root such that $internal(e) + onccyle(e) \geq \frac{n}{3}$, note that $outside(e) \leq \frac{2n}{3}$ and more importantly $internal(e) \leq \frac{2n}{3}$. Hence, the cycle corresponding to e is a $\frac{1}{3}$-$\frac{2}{3}$ separator (X, Y) for D. Indeed, such a e always exists.

- case 1: $internal(e) = 0$
- case 2: $internal(e) + oncycle(e) < internal(e') + oncycle(e') + 1 < \frac{n}{3} + 1$
- case 3: $internal(e) + oncycle(e) < internal(e') + oncycle(e') < \frac{n}{3}$
- case 4: $internal(e) + oncycle(e) = inside(e') + inside(e'') + oncycle(e') + oncycle(e'') - |p| \leq \frac{2n}{3} - |p|$

Output



- Let XY_{max} be the maximum cardinality set among X and Y. Let XY_{min} be the minimum cardinality set among X and Y. Let CE_{max} be the maximum cardinality set among C and E. Also, let CE_{min} be the minimum cardinality set among C and E. Following are the required sets:

  $A = XY_{max} \cup CE_{min}$

  $B = XY_{min} \cup CE_{max}$

  $S =$ vertices along S' unioned with M' and M''.

Outline

Other planar separator theorems of interest

- *Planar separator theorem with edge-weights*: There is a linear-time algorithm that, for a plane graph G and $\frac{1}{3}$-proper assignment [4] of nonnegative weights to edges, returns subgraphs G_1 and G_2 such that $E(G_1), E(G_2)$ is a $\frac{2}{3}$-balanced partition of $E(G)$, and $|V(G_1) \cap V(G_2)| \leq 4\sqrt{V(G)}$.

- *Planar cycle separator theorem*: There is a linear-time algorithm that, for any simple undirected biconnected triangulated plane graph and any $\frac{3}{4}$-proper assignment of nonnegative weights to faces, edges, and vertices, returns a $\frac{3}{4}$-balanced cycle separator C of size at most $4\sqrt{n}$.

— neither of these are proved in this talk

[4] an assignment is α-proper if it does not assign more than α times of the total weight of edges (resp. faces) to any edge (resp. face)