Computing an approximate minimum-degree spanning tree

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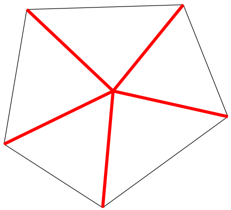
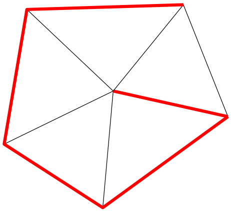
Problem description

Given an undirected graph $G(V, E)$ with $|V| = n$, find a spanning tree T of G so as to minimize the maximum degree of nodes of T.



a spanning tree



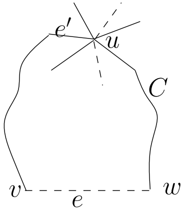
a minimum-degree spanning tree

Hardness

It is NP-hard to decide whether a given graph has a MDST of maximum degree d.

- HAMPATH $\leq_m$ MDST: a spanning tree has maximum degree two iff it is HAMPATH

Algorithm based on local search

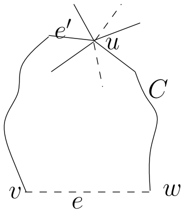


- **i** start with an arbitrary spanning tree T
- **ii** while (true)
  - **a** for some edge $e(v, w) \in G - T$, if the degree of some node u located on the unique cycle C in $T \cup \{e\}$ can be reduced

    then $T \leftarrow T \cup \{e\} - \{e'\}$ where e' incident to u and is located on C; continue

    else break
- **iii** return a *locally optimal tree LOT* (i.e., when no *local improvements* are possible)

A few constraints in choosing node u in the algorithm



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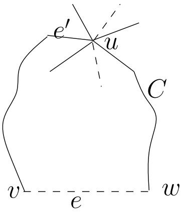

Always choose u so that it satisfies:

- $\max(d_T(v), d_T(w)) \leq d_T(u) - 2$, where $d_T(v)$ is the degree of v in T

  — otherwise, $\max(d_T(u), d_T(v), d_T(w))$ does not improve

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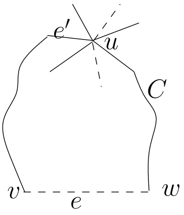

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- $d(u) \geq \Delta_T - \lg n + 1$, where $\Delta_T = \max_{u \in V} d_T(u)$

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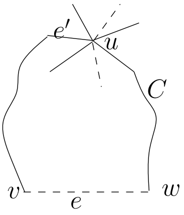

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- Let OPT be the maximum degree in an optimal tree. Next, we provide upper bounds on α, β in the following: $OPT \leq \Delta_{LOT} \leq \alpha \cdot OPT + \beta$.

- And, we upper bound the number of iterations.

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- There exists an $i \geq \Delta_{LOT} - \lg n + 1$ such that $|S_{i-1}| \leq 2|S_i|$.
  - suppose not, implying $|S_{i-1}| > 2|S_i|$ for every $i \geq \Delta_{LOT} - \lg n + 1$, leading to a contradiction: $|S_{\Delta_{LOT} - \lceil \lg n \rceil}| > 2^{\lceil \lg n \rceil} |S_{\Delta_{LOT}}| \geq n$ since $|S_{\Delta_{LOT}}| \geq 1$

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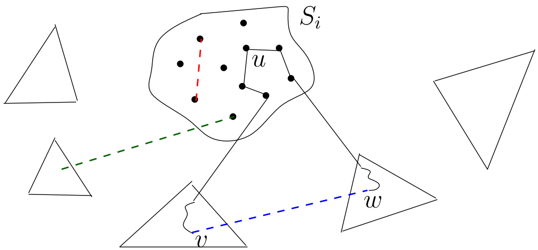
- Any spanning tree of G must have at least $|F|$ edges.

Average degree of nodes in S_{i-1} in any spanning tree



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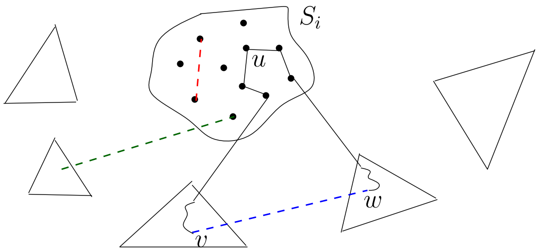

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- Due to local optimality of LOT, every edge in $G - LOT$ that connects any two components of $LOT - S_i$ is incident to some node in S_{i-1}.

- Therefore,
  OPT
  $\geq$ average degree of nodes in S_{i-1} in any spanning tree of G is
  $\geq \frac{|F|+|S_i|-1}{|S_{i-1}|}$.

Upper bounding the approximation factor

- $OPT \geq \frac{|F|+|S_i|-1}{|S_{i-1}|} \geq \frac{(i-1)|S_i|+1}{2|S_i|} > \frac{(i-1)}{2}$.

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  Since $i \geq \Delta_{LOT} - \lg n + 1,\ \Delta_{LOT} \leq 2OPT + \lg_2 n$.

Time complexity

Let $\phi(T) = \sum_{v \in V} 3^{d_T(v)}$ be the potential of any tree T.

- The initial potential of any tree is at most $n3^{\Delta_T}$.
  The lowest potential is for a HAMPATH, and it is $(2)(3) + (n-2)3^2 > n$.

[1] since $i \geq \Delta_T - \lceil \lg n \rceil + 1$

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- After an iteration of improvement to any tree T, resulting in a tree T' by including edge (v, w) and reducing the degree of u from i to $i-1$, $\phi(T') \leq (1 - \frac{2}{27n^3})\phi(T)$.

  decrease in potential = decrease in potential due to degree change at u − maximum increase in potential due to increase in degrees of v and w $= (3^i - 3^{i-1}) - (2(3^{i-1} - 3^{i-2})) = (2)3^{i-1} - (4)3^{i-2} > \frac{2}{9}3^i \geq \frac{2}{9}3^{\Delta_T - (\lg n) + 1}{}^{1} = \frac{2}{27}\frac{1}{3^{(\lg n) - 2}}3^{\Delta_T} \geq \frac{2}{27n^2}3^{\Delta_T} \geq \frac{2}{27n^3}\phi(T)$

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  Since the lowest possible potential is $> n$, the number of local moves is $O(n^4)$.

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polynomial time $(OPT + 1)$-apprx algo is known for this problem

— not presented

Open problems

- improving apprx factor of directed version of the problem
- d-vertex/edge-connected subgraph with minimum degree
- minimum degree spanners
- multicriteria optimization: maximum number of leaves and minimum degree
- relation among problems that have no multiplicative error but have small additive error

References

M. Furer and B. Raghavachari. Approximating the minimum-degree spanning tree to within one from the optimal degree. SODA, 1992.

David P. Williamson and David B. Shmoys. The Design of Approximation Algorithms. Cambridge University Press, 2011.

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Thanks!