- The recurrence relation for *Fibonacci numbers* is,

$$f_i = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ f_{i-1} + f_{i-2} & \text{if } i \geq 2. \end{cases}$$

- The ordinary generating function of this sequence is $G(x) = f_0 x^0 + f_1 x^1 + \ldots$.

  Since $f_i + f_{i+1} = f_{i+2}$ for $i \geq 2$,
  $G(x) - xG(x) - x^2 G(x) = x$, i.e., $G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\alpha x)(1-\beta x)}$ for $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

  The $\frac{1+\sqrt{5}}{2}$ is known as the *golden ratio*, denoted by ϕ. Naturally, $\frac{1-\sqrt{5}}{2}$ is known as the *conjugate of the golden ratio*, denoted by $\widehat{\phi}$.

  Specifically, $\frac{x}{(1-\phi x)(1-\widehat{\phi} x)} = \frac{1}{\sqrt{5}}(\frac{1}{1-\phi x} - \frac{1}{1-\widehat{\phi} x}) = \frac{1}{\sqrt{5}}((1+(\phi x)+(\phi x)^2+\ldots)-(1+(\widehat{\phi} x)+(\widehat{\phi} x)^2+\ldots))$.
  Therefore, $f_n = [x^n]G(x) = \frac{1}{\sqrt{5}}(\phi^n - \widehat{\phi}^n) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$.

- Aliter:

  Since $f_n = f_{n-1} + f_{n-2}$ is a linear homogenous recurrence relation with degree two, its solution must be of form $\alpha_1 r_1^n + \alpha_2 r_2^n$ when $r_1 \neq r_2$.
  Here, α_1 and α_2 are real numbers, r_1 and r_2 are the roots of the characteristic equation $r^2 - r - 1 = 0$ of the recurrence relation. Solving the characteristic equation yields $r_1 = \phi$ and $r_2 = \widehat{\phi}$.
  Since $f_0 = 0$ and $f_1 = 1$, $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 \phi + \alpha_2 \widehat{\phi} = 1$. Solving these two equations, $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$.
  Therefore, $f_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\widehat{\phi}^n$.

- Observation: For any n, f_{n+1} is the number of ordered multisets consisting of 1s and 2s such that the sum of elements of each such set is equal to n. For example, since 4 can be written as $1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2$, f_5 is equal to five.

- Significant (in)equalities:

  $\sum_{i=0}^{n} f_i = f_{n+2} - 1$
  $\sum_{i=0}^{n-1} f_{2i+1} = f_{2n}$
  $\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$ $\rightarrow$ leading to Fibonacci spiral

[1] with the help of note taken by Sawinder Kaur (TA) in a lecture

$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ ← Cassini's theorem
$\phi^n = \phi f_n + f_{n-1}$
$f_n \geq \phi^{n-2}$

- The recurrence relation for *Catalan numbers* is,

$$C_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ \sum_{i=1}^{n-1} C_i C_{n-i} & \text{if } n \geq 2. \end{cases}$$

The ordinary generating function of this sequence is $G(x) = C_0x^0 + C_1x^1 + C_2x^2 + \ldots$.

Then, $G(x) - x$
$= C_2x^2 + C_3x^3 + \ldots$
$= \sum_{n\geq 2} C_n x^n$
$= \sum_{n\geq 2} \sum_{i=1}^{n-1} C_i C_{n-i} x^n$
$= \sum_{n\geq 2} \sum_{i=1}^{n-1} C_i x^i C_{n-i} x^{n-i}$
$= (\sum_{i\geq 1} C_i x^i)(\sum_{j\geq 1} C_j x^j)$
$= (G(x))^2$.

Therefore, $G(x)^2 - G(x) + x = 0$, i.e., $G(x) = \frac{1\pm\sqrt{1-4x}}{2}$.

If $G(x) = \frac{1+\sqrt{1-4x}}{2}$, $G(0) = 1$; however, $C_0 = 0$. Hence, $G(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}$.

Using extended binomial theorem, $G(x) = \frac{1}{2} - \frac{1}{2}(\sum_{k\geq 0} \binom{1/2}{k}(-4x)^k)$.

Therefore, $[x^n]G(x) = \frac{-1}{2}\binom{1/2}{n}(-4)^n$.

But, $\binom{1/2}{n}$
$= \frac{(1/2)(1/2-1)(1/2-2)...(1/2-(n-1))}{n!}$
$= \frac{(1/2)(-1/2)(-3/2)...(-(\frac{2n-3}{2}))}{n!}$
$= \frac{1}{2^n}\frac{(-1)^n(1)(3)...(2n-3)}{n!}$
$= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n!}\frac{(2n-2)!}{(2)(4)...(2n-2)}$
$= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n!}\frac{(2n-2)!}{2^{n-1}(n-1)!}$
$= \frac{2}{4^n}(-1)^{n-1}\frac{(2n-2)!}{n!(n-1)!}$
$= \frac{2}{4^n n}(-1)^{n-1}\binom{2n-2}{n-1}$.

Hence, $[x^n]G(x) = \frac{-1}{2}\frac{2}{4^n n}(-1)^{n-1}\binom{2n-2}{n-1}(-1)^n 4^n = \frac{(-1)^{2n}}{n}\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

If the recurrence is defined as,

$$C_n = \begin{cases} 1 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ \sum_{i=1}^{n-1} C_i C_{n-i} & \text{if } n \geq 2. \end{cases}$$

substituting $n+1$ for n in the above, $C_n = [x^n]G(x) = \frac{1}{n+1}\binom{2n}{n}$.

Due to Stirling, $\sqrt{2\pi n}(\frac{n}{e})^n e^{(\frac{1}{12n}-\frac{1}{360n^3})} < n! < \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}$; from these inequalities, C_n is $\Omega(\frac{4^n}{n^{3/2}})$: $\frac{1}{n+1}\frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{2\pi n(\frac{n}{e})^{2n}} = \frac{1}{n+1}\frac{4^n}{\sqrt{\pi n}} = \frac{4^n}{n^{3/2}}(\frac{1}{\sqrt{\pi}}(1-\frac{1}{1+n}))$.

- A function $f(n)$ is *monotonically increasing (resp. monotonically decreasing)* if $m \leq n$ implies $f(m) \leq f(n)$ (resp. $f(m) \geq f(n)$). A function $f(n)$ is strictly increasing (resp. strictly decreasing) if $m < n$ implies $f(m) < f(n)$ (resp. $f(m) > f(n)$).

  Let $f(x)$ be a positive monotonically increasing continuous function. By approximating the area under $f(x)$ by two step functions, the following inequalities are derived.

  * $\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$.

    (The left (resp. right) ineq is due to the left (resp. right) subfigure of Fig. 1.)

  * $f(m) + \int_{m}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq f(n) + \int_{m}^{n} f(x)dx$.

    (The left (resp. right) ineq is due to the left (resp. right) subfigure of Fig. 1.)

  The second one is preferred as it does not rely on the integral values outside $[m, n]$.

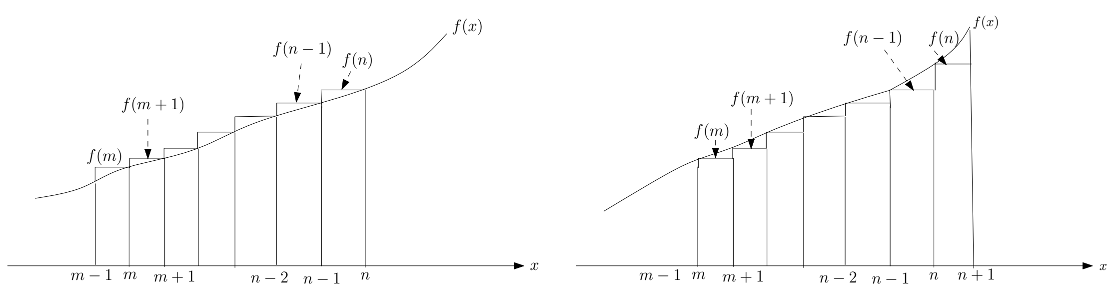

Figure 1: Approximating a summation with an integral.

- An example: Noting that $\lg(k)$ is a monotonically increasing continuous function, $n\ln(n) - n + 1 \leq \sum_{i=1}^{n} \ln(i) \leq n\ln(n) - n + 1 + \ln(n)$. ——— (1)

  Hence, $\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$. ——— (1b)

- From (1), we know $\lg(n!) - n\ln(n) + n \leq 1 + \ln(n)$.
  We claim there exists a positive constant α such that $(\ln(n!) - n\ln n + n - \frac{1}{2}\ln n) \approx \alpha$.
  Then, $e^{\alpha} \approx e^{(\ln(n!) - (n+\frac{1}{2})\ln n + n)} = \frac{n!e^n}{n^{n+1/2}}$. ——- (2a)
  That is, $n! \approx e^{\alpha} n^{n+\frac{1}{2}} e^{-n}$. —— (2b)

  From Wallis' inequality, when n is asymptotically large, $\frac{(2)(4)(6)...(2n)}{(1)(3)(5)...(2n-1)\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}}$
  $\Rightarrow \frac{(2^n n!)^2}{(2n)!} \frac{1}{\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}}$.

  Substituting (2b), $\frac{2^{2n} e^{2\alpha} n^{2n+1} e^{-2n}}{e^{\alpha}(2n)^{2n+\frac{1}{2}} e^{-2n}} \frac{1}{\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}} \Rightarrow e^{\alpha} \approx \sqrt{2\pi}$.

  Substituting (2a), $e^{\alpha} = \frac{n!e^n}{n^{n+1/2}} \approx \sqrt{2\pi} \Rightarrow n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$.

  This is known as the *Stirling's approximation of* $n!$. As n grows, Stirling's approximation betters in approximating $n!$ to (1b).

- The recurrence relation for *Harmonic numbers* is,

$$H_n = \begin{cases} 1 & \text{if } n = 1, \\ a_{n-1} + \frac{1}{n} & \text{if } n \geq 2. \end{cases}$$

  If $f(k)$ is a positive monotonically decreasing continuous function, then

  - $\int_m^{n+1} f(x)dx \leq \sum_{k=m}^n f(k) \leq \int_{m-1}^n f(x)dx$
  - $f(n) + \int_m^n f(x)dx \leq \sum_{k=m}^n f(k) \leq f(m) + \int_m^n f(x)dx$ —— (3)

  Noting that $1/k$ is a monotonically decreasing function, from (3), $\frac{1}{n} + \ln(n) \leq \sum_{k=1}^n \frac{1}{k} \leq 1 + \ln(n)$. —— (4)

* Aliter: Split $[1, n]$ into $\lfloor \lg n \rfloor + 1$ pieces and upper-bound the contribution of each piece by 1:

$$\begin{aligned} H_n = \sum_{k=1}^n \frac{1}{k} &\leq \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \\ &\leq \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i - 1} \frac{1}{2^i} \\ &= \sum_{i=0}^{\lfloor \lg n \rfloor} \frac{1}{2^i}(2^i - 1 + 1) \\ &= \lg n + 1 \end{aligned}$$

- The *sum of first* n *Harmonic numbers* is, $\sum_{k=1}^{n} H_k$
$= \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{j}$
$= 1 + (1 + \frac{1}{2}) + (1 + \frac{1}{2} + \frac{1}{3}) + \ldots + (1 + \frac{1}{2} + \ldots + \frac{1}{n})$
$= n + \frac{1}{2}(n-1) + \frac{1}{3}(n-2) + \ldots + \frac{1}{n}(n-(n-1))$
$= \sum_{j=1}^{n} \frac{1}{j}(n-j+1)$
$= \sum_{j=1}^{n} (\frac{n+1}{j} - \frac{j}{j})$
$= (n+1)(\sum_{j=1}^{n} \frac{1}{j}) - (\sum_{j=1}^{n} 1)$
$= (n+1)H_n - n.$

- An application: We had seen two proofs, one by Euclid and the other by Christian Goldbach, to show there are *infinitely many primes*. Here is another proof of the same by Leonhard Euler. Let n be a positive integer. Also, let $\pi(n)$ be the number of primes less than or equal to n. Below, p denotes a prime number.
$\ln n \leq^{\text{(from the left ineq in (4))}} \sum_{k=1}^{n} \frac{1}{k} \leq^{(a)\text{, see below}} \Pi_{p \leq n}(\sum_{k \geq 0} \frac{1}{p^k}) = \Pi_{p \leq n}(\frac{1}{1-\frac{1}{p}}) = \Pi_{k=1}^{\pi(n)} \frac{p_k}{p_k - 1} \leq \Pi_{k=1}^{\pi(n)}(1 + \frac{1}{p_k - 1}) \leq^{(\text{since } p_k \geq k+1)} \Pi_{k=1}^{\pi(n)}(1 + \frac{1}{k}) = \Pi_{k=1}^{\pi(n)}(\frac{k+1}{k}) = \pi(n) + 1.$
For every integer i, if i has a prime decomposition comprising a subset of primes that are less than or equal to n, then $\frac{1}{i}$ contributes to the RHS of inequality (a).
Since $\ln n$ strictly monotonically increases with n, $\pi(n)$ must also increase with n.

References:

- Discrete Mathematics and its Applications by K. H. Rosen, Eighth Edition.

- Introduction to Algorithms by T. H. Cormen, C. E. Leiserson, R. L. Riverst, and C. Stein, Third Edition.

- Plus, wiki and the folklore.