- The recurrence relation for *Fibonacci numbers* is,

$$f_i = \begin{cases} 0 \ \text{ if } i = 0, \\ 1 \ \text{ if } i = 1, \\ f_{i-1} + f_{i-2} \ \text{ if } i \geq 2. \end{cases}$$

- The ordinary generating function of this sequence is $G(x) = f_0 x^0 + f_1 x^1 + \ldots$.

  Since $f_i + f_{i+1} = f_{i+2}$ for $i \geq 2$,
  $G(x) - xG(x) - x^2 G(x) = x$, i.e., $G(x) = \frac{x}{1-x-x^2} = \frac{x}{(1-\alpha x)(1-\beta x)}$ for $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

  The $\frac{1+\sqrt{5}}{2}$ is known as the *golden ratio*, denoted with $\phi$. Naturally, $\frac{1-\sqrt{5}}{2}$ is known as the *conjugate of the golden ratio*, denoted with $\widehat{\phi}$.

  Specifically, $\frac{x}{(1-\phi x)(1-\widehat{\phi} x)} = \frac{1}{\sqrt{5}}(\frac{1}{1-\phi x} - \frac{1}{1-\widehat{\phi} x}) = \frac{1}{\sqrt{5}}((1+(\phi x)+(\phi x)^2+\ldots)-(1+(\widehat{\phi} x)+(\widehat{\phi} x)^2+\ldots))$.
  Therefore, $f_n = [x^n]G(x) = \frac{1}{\sqrt{5}}(\phi^n - \widehat{\phi}^n) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$.

- Aliter:

  Since $f_n = f_{n-1} + f_{n-2}$ is a linear homogenous recurrence relation with degree two, its solution must be of form $\alpha_1 r_1^n + \alpha_2 r_2^n$ when $r_1 \neq r_2$.
  Here, $\alpha_1$ and $\alpha_2$ are real numbers, $r_1$ and $r_2$ are the roots of the characteristic equation $r^2 - r - 1 = 0$ of the recurrence relation. Solving the characteristic equation yields $r_1 = \phi$ and $r_2 = \widehat{\phi}$.
  Since $f_0 = 0$ and $f_1 = 1$, $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 \phi + \alpha_2 \widehat{\phi} = 1$. Solving these two equations, $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$.
  Therefore, $f_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\widehat{\phi}^n$.

- Observation: For any $n$, $f_{n+1}$ is the number of ordered multisets consisting of 1s and 2s such that the sum of elements of each such set is equal to $n$. For example, since $4$ can be written as $1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2$, $f_5$ is equal to five.

- Significant (in)equalities:

  $\sum_{i=0}^{n} f_i = f_{n+2} - 1$
  $\sum_{i=0}^{n-1} f_{2i+1} = f_{2n}$
  $\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$ $\rightarrow$ leading to Fibonacci spiral

[1] with the help of note taken by Sawinder Kaur (TA) in a lecture

$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ ← Cassini's theorem
$\phi^n = \phi f_n + f_{n-1}$
$f_n \geq \phi^{n-2}$

- The recurrence relation for *Catalan numbers* is,

$$C_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ \sum_{i=1}^{n-1} C_i C_{n-i} & \text{if } n \geq 2. \end{cases}$$

The ordinary generating function of this sequence is $G(x) = C_0x^0 + C_1x^1 + C_2x^2 + \ldots$.

Then, $G(x) - x$
$= C_2x^2 + C_3x^3 + \ldots$
$= \sum_{n\geq 2} C_n x^n$
$= \sum_{n\geq 2} \sum_{i=1}^{n-1} C_i C_{n-i} x^n$
$= \sum_{n\geq 2} \sum_{i=1}^{n-1} C_i x^i C_{n-i} x^{n-i}$
$= (\sum_{i\geq 1} C_i x^i)(\sum_{j\geq 1} C_j x^j)$
$= (G(x))^2$.

Therefore, $G(x)^2 - G(x) + x = 0$, i.e., $G(x) = \frac{1\pm\sqrt{1-4x}}{2}$.

If $G(x) = \frac{1+\sqrt{1-4x}}{2}$, $G(0) = 1$; however, $C_0 = 0$. Hence, $G(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}$.

Using extended binomial theorem, $G(x) = \frac{1}{2} - \frac{1}{2}(\sum_{k\geq 0} \binom{1/2}{k}(-4x)^k)$.

Therefore, $[x^n]G(x) = \frac{-1}{2}\binom{1/2}{n}(-4)^n$.

But, $\binom{1/2}{n}$
$= \frac{(1/2)(1/2-1)(1/2-2)...(1/2-(n-1))}{n!}$
$= \frac{(1/2)(-1/2)(-3/2)...(-(\frac{2n-3}{2}))}{n!}$
$= \frac{1}{2^n}\frac{(-1)^n(1)(3)...(2n-3)}{n!}$
$= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n!}\frac{(2n-2)!}{(2)(4)...(2n-2)}$
$= \frac{1}{2^n}(-1)^{n-1}\frac{1}{n!}\frac{(2n-2)!}{2^{n-1}(n-1)!}$
$= \frac{2}{4^n}(-1)^{n-1}\frac{(2n-2)!}{n!(n-1)!}$
$= \frac{2}{4^n n}(-1)^{n-1}\binom{2n-2}{n-1}$.

Hence, $[x^n]G(x) = \frac{-1}{2}\frac{2}{4^n n}(-1)^{n-1}\binom{2n-2}{n-1}(-1)^n 4^n = \frac{(-1)^{2n}}{n}\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

If the recurrence is defined as,

$$C_n = \begin{cases} 1 \text{ if } n = 0, \\ 2 \text{ if } n = 1, \\ \sum_{i=1}^{n-1} C_i C_{n-i} \text{ if } n \geq 2. \end{cases}$$

substituting $n+1$ for $n$ in the above, $C_n = [x^n]G(x) = \frac{1}{n+1}\binom{2n}{n}$.

Due to Stirling, $\sqrt{2\pi n}(\frac{n}{e})^n e^{(\frac{1}{12n} - \frac{1}{360n^3})} < n! < \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}}$; from these inequalities, $C_n$ is $\Omega(\frac{4^n}{n^{3/2}})$: $\frac{1}{n+1}\frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{2\pi n(\frac{n}{e})^{2n}} = \frac{1}{n+1}\frac{4^n}{\sqrt{\pi n}} = \frac{4^n}{n^{3/2}}(\frac{1}{\sqrt{\pi}}(1 - \frac{1}{1+n}))$.

- A function $f(n)$ is *monotonically increasing (resp. monotonically decreasing)* if $m \leq n$ implies $f(m) \leq f(n)$ (resp. $f(m) \geq f(n)$). A function $f(n)$ is strictly increasing (resp. strictly decreasing) if $m < n$ implies $f(m) < f(n)$ (resp. $f(m) > f(n)$).

  Let $f(x)$ be a positive monotonically increasing continuous function. By approximating the area under $f(x)$ by two step functions, the following inequalities are derived.

  * $\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$.

    (The left (resp. right) ineq is due to the left (resp. right) subfigure of Fig. 1.)

  * $f(m) + \int_{m}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq f(n) + \int_{m}^{n} f(x)dx$.

    (The left (resp. right) ineq is due to the left (resp. right) subfigure of Fig. 1.)

  The second one is preferred as it does not rely on the integral values outside $[m, n]$.

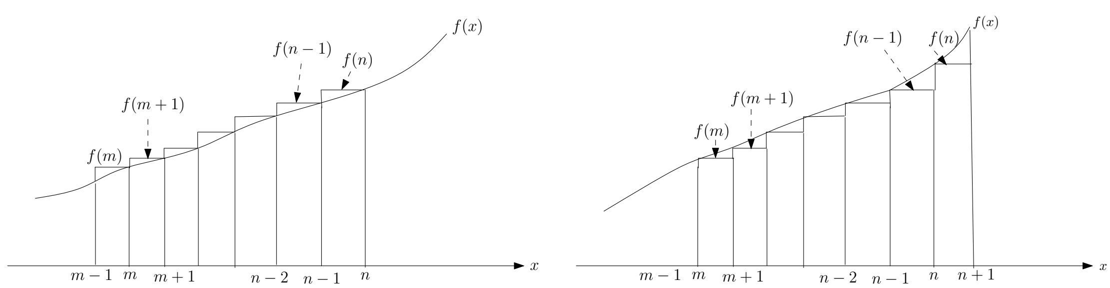


Figure 1: Approximating a summation with an integral.

- An example: Noting that $\lg(k)$ is a monotonically increasing continuous function, $n\ln(n) - n + 1 \leq \sum_{i=1}^{n} \ln(i) \leq n\ln(n) - n + 1 + \ln(n)$. ——— (1)

  Hence, $\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$. ——— (1b)

- From (1), we know $\lg(n!) - n\ln(n) + n \leq 1 + \ln(n)$.
  We claim there exists a positive constant $\alpha$ such that $(\ln(n!) - n\ln n + n - \frac{1}{2}\ln n) \approx \alpha$.
  Then, $e^{\alpha} \approx e^{(\ln(n!) - (n+\frac{1}{2})\ln n + n)} = \frac{n! e^n}{n^{n+1/2}}$. ——- (2a)
  That is, $n! \approx e^{\alpha} n^{n+\frac{1}{2}} e^{-n}$. —— (2b)

  From Wallis' inequality, when $n$ is asymptotically large, $\frac{(2)(4)(6)...(2n)}{(1)(3)(5)...(2n-1)\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}}$
  $\Rightarrow \frac{(2^n n!)^2}{(2n)!} \frac{1}{\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}}$.

  Substituting (2b), $\frac{2^{2n} e^{2\alpha} n^{2n+1} e^{-2n}}{e^{\alpha} (2n)^{2n+\frac{1}{2}} e^{-2n}} \frac{1}{\sqrt{2n}} \approx \sqrt{\frac{\pi}{2}} \Rightarrow e^{\alpha} \approx \sqrt{2\pi}$.

  Substituting (2a), $e^{\alpha} = \frac{n! e^n}{n^{n+1/2}} \approx \sqrt{2\pi} \Rightarrow n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$.

  This is known as the *Stirling's approximation of* $n!$. As $n$ grows, Stirling's approximation betters in approximating $n!$ to (1b).

- The recurrence relation for *Harmonic numbers* is,

$$H_n = \begin{cases} 1 \text{ if } n = 1, \\ a_{n-1} + \frac{1}{n} \text{ if } n \geq 2. \end{cases}$$

  If $f(k)$ is a positive monotonically decreasing continuous function, then

  - $\int_m^{n+1} f(x)dx \leq \sum_{k=m}^n f(k) \leq \int_{m-1}^n f(x)dx$
  - $f(n) + \int_m^n f(x)dx \leq \sum_{k=m}^n f(k) \leq f(m) + \int_m^n f(x)dx$ —— (3)

  Noting that $1/k$ is a monotonically decreasing function, from (3), $\frac{1}{n} + \ln(n) \leq \sum_{k=1}^n \frac{1}{k} \leq 1 + \ln(n)$. —— (4)

* Aliter: Split $[1, n]$ into $\lfloor \lg n \rfloor + 1$ pieces and upper-bound the contribution of each piece by 1:

$$\begin{aligned} H_n = \sum_{k=1}^n \frac{1}{k} &\leq \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \\ &\leq \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i - 1} \frac{1}{2^i} \\ &= \sum_{i=0}^{\lfloor \lg n \rfloor} \frac{1}{2^i}(2^i - 1 + 1) \\ &= \lg n + 1 \end{aligned}$$

- The *sum of first $n$ Harmonic numbers* is, $\sum_{k=1}^{n} H_k$
$= \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{j}$
$= 1 + (1 + \frac{1}{2}) + (1 + \frac{1}{2} + \frac{1}{3}) + \ldots + (1 + \frac{1}{2} + \ldots + \frac{1}{n})$
$= n + \frac{1}{2}(n-1) + \frac{1}{3}(n-2) + \ldots + \frac{1}{n}(n-(n-1))$
$= \sum_{j=1}^{n} \frac{1}{j}(n-j+1)$
$= \sum_{j=1}^{n} (\frac{n+1}{j} - \frac{j}{j})$
$= (n+1)(\sum_{j=1}^{n} \frac{1}{j}) - (\sum_{j=1}^{n} 1)$
$= (n+1)H_n - n.$

- We had seen two proofs, one by Euclid and the other by Christian Goldbach, to show there are *infinitely many primes*. Here is another proof of the same by Leonhard Euler. Let $n$ be a positive integer. Also, let $\pi(n)$ be the number of primes less than or equal to $n$. Below, $p$ denotes a prime number.

  $\ln n \leq^{\text{(from the left ineq in (4))}} \sum_{k=1}^{n} \frac{1}{k} \leq^{(a)\text{, see below}} \Pi_{p \leq n}(\sum_{k \geq 0} \frac{1}{p^k}) = \Pi_{p \leq n}(\frac{1}{1-\frac{1}{p}}) = \Pi_{k=1}^{\pi(n)} \frac{p_k}{p_k - 1} \leq \Pi_{k=1}^{\pi(n)}(1 + \frac{1}{p_k - 1}) \leq^{(\text{since } p_k \geq k+1)} \Pi_{k=1}^{\pi(n)}(1 + \frac{1}{k}) = \Pi_{k=1}^{\pi(n)}(\frac{k+1}{k}) = \pi(n) + 1.$

  For every integer $i$, if $i$ has a prime decomposition comprising a subset of primes that are less than or equal to $n$, then $\frac{1}{i}$ contributes to the RHS of inequality $(a)$.

  Since $\ln n$ strictly monotonically increases with $n$, $\pi(n)$ must also increase with $n$.

- The number of ways to distribute $n$ labeled balls into $r$ unlabeled (indistinguishable) bins with no bin left empty is $\frac{1}{r!}\cdot$ (number of onto functions from a set with $n$ elements to a set with $r$ elements) $= \frac{1}{r!}\sum_{i=0}^{r-1}(-1)^i C(r,i)(r-i)^n$. This is called the $r^{th}$ *Stirling number of the second kind of* $n$, denoted by $\left\{ {n \atop r} \right\}$. For asymptotically large values of $n$, $\left\{ {n \atop r} \right\}$ is approximately $\frac{r^n}{r!}$.[2]

  The recurrence relation for distributing $n+1$ labeled balls into $r$ unlabeled bins with no bin left empty is, $\left\{ {n+1 \atop r} \right\} = r\left\{ {n \atop r} \right\} + \left\{ {n \atop r-1} \right\}$ for $0 < r < n$, with initial conditions $\left\{ {0 \atop 0} \right\} = 1, \left\{ {k \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 0$ for $k > 0$: In partitioning the $n+1$ labeled balls into $r$ bins such that no bin left empty, the $(n+1)$-th ball could either be in a bin with no other ball (case (i)) or it shares the bin with other balls (case (ii)). In case (i), the remaining $n$ balls needs to be partitioned into $r-1$ bins, hence the second term of the recurrence. In case (ii), all balls other than the $(n+1)$-th ball are partitioned into $r$ bins, and then there are $r$ choices for inserting the $(n+1)$-th ball, hence the first term.

  From the definition of $\left\{ {n \atop j} \right\}$, the number of ways to distribute $n$ labeled balls into at most $r$ unlabeled bins is $\sum_{j=1}^{r}\left\{ {n \atop j} \right\}$. When $r$ is $n$, this is denoted by $B_n$. For a set $S = \{a,b,c\}$ of (labeled) balls, $B_3 = 5$: $\{\{a\},\{b\},\{c\}\}, \{\{a\},\{b,c\}\}, \{\{b\},\{a,c\}\}, \{\{c\},\{a,b\}\}, \{\{a,b,c\}\}$. The $B_n$ is known to be

[2] not proved in class

$O(\frac{n^n}{\ln{(n+1)}})$.[3]
Considering the combinatorial meaning of $\left\{ {n \atop r} \right\}$, $B_n = \sum_{r=0}^{n} \left\{ {n \atop r} \right\}$.
Further, $B_{n+1} = \sum_{r=0}^{n} C(n,r) B_r$: fix on an arbitrary labeled ball $b$ in $n+1$ balls; choose $r$ balls from $n$ balls in $C(n,r)$ ways, partition those $r$ balls in $B_r$ ways, and include the partition that contains the remaining $n-r$ balls and the ball $b$.

- A partition of a positive integer $n$, also called an *integer partition*, is a way of writing $n$ as a sum of positive integers. For example, for $n = 4$, there are five distinct integer partitions: $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. The number of integer partitions of $n$ is denoted by $p(n)$. For every positive integer $i$, let $a_i$ be the number of times $i$ occurs in an integer partition of $n$. Then the number of non-negative integral solutions of $(1)a_1 + (2)a_2 + \ldots + k(a_k) + \ldots = n$ is the number of partitions of $n$, which is essentially $p(n)$. Therefore, $p(n) = [x^n]((1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)\ldots) = [x^n](\Pi_{i=1}^{\infty} \frac{1}{1-x^k})$. No closed formula is known for $p(n)$. However, it is known that as $n \to \infty$, $p(n)$ is approximately equal to $\frac{1}{4n\sqrt{3}} e^{(\pi \sqrt{\frac{2n}{3}})}$.

  The number of ways of partitioning $n$ in exactly $r$ positive integers is denoted by $p_r(n)$. The $p_r(n)$ is also the number of ways to distribute $n$ unlabeled balls into $r$ unlabeled bins with no bin left empty. Finding $p_r(n)$ is equivalent to finding the number of monotonically increasing positive integers $a_1, a_2, \ldots, a_r$ such that $a_1 + a_2 + \ldots + a_r = n$, which is equal to $[x^n] \frac{x^r}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)}$.
  The recurrence relation is, $p_r(n) = p_{r-1}(n-1) + p_r(n-r)$ with $p_k(n) = 0$ for $k > n$, $p_1(n) = p_n(n) = 1, p_2(n) = \lfloor \frac{n}{2} \rfloor$, and $p_0(n) = 0$: when there is a partition of size one, partition the remaining $n-1$ unlabelled balls in $r-1$ parts; when there is no partition of size one, keep aside $r$ balls, partition the remaining $n-r$ balls in $r$ parts, and add one ball in each part so that each part will be of size at least two.

  A partition of a positive integer $n$ into at most $r$ positive integers is a way of writing $n$ as a sum of at most $r$ positive integers. When $n = 4$ and $r = 3$, there are four distinct integer partitions: $4, 3+1, 2+2, 2+1+1$. Analogously, there are nine ways of distributing 6 identical books into 4 identical boxes: $6$, $5+1$, $4+2$, $4+1+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$. This is equal to $\sum_{i=1}^{r} p_i(n)$.


References:

- Discrete Mathematics and its Applications by K. H. Rosen, Eighth Edition.

- Introduction to Algorithms by T. H. Cormen, C. E. Leiserson, R. L. Riverst, and C. Stein, Third Edition.

- Plus, wiki and the folklore.


[3] not proved in class