- Given a set of n points, any subset of k points that can be separated from the other $n-k$ points by a hyperplane is called a *k-set*.

  For the planar case, the upper bound on the number of k-sets in a set of n points is $O(nk^{1/3})$. [Dey '98]

  - First note the following: For a given set of n points in the plane in general position, a pair of points $p, q \in P$ form a k-set if there are exactly k points in the (closed) halfplane below the line passing through p and q.
  - Consider the graph $G(P, E)$ that has an edge for every k-set

    edges of G can be decomposed into $k-1$ (resp. $n-k+1$) convex chains

    any crossing of two edges of G is an intersection point of one convex chain of $C_1, \ldots, C_{k-1}$ with a concave chain of $D_1, \ldots, D_{n-k+1}$; hence, there are at most $2(k-1)(n-k+1)$ crossings of $G \Rightarrow m^3/n^2 = O(nk)$, which implies $m = O(nk^{1/3})$

  The maximum numebr of k-sets in a set of n points located in $\mathbb{R}^d$ is still unknown.

- The *k-level* in an arrangement $A(H)$ induced by a set H of n hyperplanes is defined as the set of points with at most $k-1$ hyperplanes strictly above it, and at most $n-k$ hyperplanes strictly below it.

  For the planar case, the upper bound on the complexity of any k-level in an arrangement of n lines is $O(nk^{1/3})$.

  - since the k-level problem is the k-set problem in the dual setting

  The tight bounds on the maximum complexity of any k-level is still unknown, even in the planar case.

- Assuming lines are in general position, the number of vertices of level *at most* k in an arrangement $A(H)$ of set H of n lines in the plane is $O(nk)$.

  * choose a subset $R \subseteq H$ at random, by including each line $h \in H$ into R with probability p; let f denote the number of vertices of level 0 in the arrangement $A(R)$; $E[f] \leq E[|R|] = pn$
  * for any vertex v of $A(H)$, let A_v be an event of v becoming one of the vertices of level 0 in $A(R)$; $p[A_v] = p^2(1-p)^{\ell(v)}$, where $\ell(v)$ denotes the level of v in $A(H)$;
  * let $V_{\leq k}$ (resp. V) be the set of vertices of level at most k (resp. 0) in $A(H)$;

    $np \geq E[f] = \sum_{v \in V} prob(A_v) \geq \sum_{v \in V_{\leq k}} prob(A_v) = \sum_{v \in V_{\leq k}} p^2(1-p)^{\ell(v)} \geq |V_{\leq k}| p^2 (1-p)^k \Rightarrow |V_{\leq k}| \leq \frac{n}{p(1-p)^k}$; choosing $p = \frac{1}{k+1}$ to (approximately) maximize RHS gives $|V_{\leq k}| \leq 3(k+1)n$

  Clarkson-Shor theorem: The number of vertices of level *at most* k in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $O(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$.

  — proof is a generalization of the above argument

- Given a set of n points in the plane, any subset of at most k points that can be separated from all other point (there are at least $n-k$ of them) by a line is called a *$(\leq k)$-set*.

  In the plane, the maximum number of $(\leq k)$-sets is $\Theta(nk)$, and it is $\Theta(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$ in $\mathbb{R}^d$.

References:
- Lectures on Discrete Geometry by J. Matousek.