

Principle: There is no infinite sequence of strictly decreasing non-negative integers.

To prove a predicate $P(k)$ is true for every $k \in \mathbb{Z}^{0+}$, on the way of arriving to a contradiction, suppose $\neg P(k)$ is true for some $k \in \mathbb{Z}^{0+}$. Construct a solution k_1 to $\neg P(k)$, prove the existence of k_1 implies another solution k_2 to $\neg P(k)$, \dots , wherein k_1, k_2, \dots , is a decremental sequence of non-negative integers. As you might have noted, this proof method is closely related to proving using well-ordering principle.

- For a positive integer k , \sqrt{k} is irrational if it is not an integer.

$$\sqrt{k} = \frac{m}{n} = \frac{m(\sqrt{k} - \lfloor \sqrt{k} \rfloor)}{n(\sqrt{k} - \lfloor \sqrt{k} \rfloor)} = \frac{m'}{n'}$$

since $0 < (\sqrt{k} - \lfloor \sqrt{k} \rfloor) < 1$, $m' < m$, $n' < n$, and m', n' are positive

$$\text{further, since } \frac{m'}{n'} = \frac{m(\sqrt{k} - \lfloor \sqrt{k} \rfloor)}{n(\sqrt{k} - \lfloor \sqrt{k} \rfloor)} = \frac{nk - m\lfloor \sqrt{k} \rfloor}{m - n\lfloor \sqrt{k} \rfloor}, m' \text{ and } n' \text{ are integers}$$

- No integral solutions to $a^2 + b^2 = 3(s^2 + t^2)$, other than the trivial solution $a = b = s = t = 0$.

noting that $(3|(a_1^2 + b_1^2))$ iff $(3|a_1 \text{ and } 3|b_1)$,

- proof of \Leftarrow is immediate; for \Rightarrow , consider the contrapositive: $((3 \nmid a) \vee (3 \nmid b)) \Rightarrow (3 \nmid (a^2 + b^2)) \equiv ((3 \nmid a) \Rightarrow (3 \nmid (a^2 + b^2))) \wedge ((3 \nmid b) \Rightarrow (3 \nmid (a^2 + b^2)))$
- in proving the former, $(3 \nmid a) \Rightarrow (a = 3i + r)$ where $r \in \{1, 2\}$ and $b = 3j + r'$ where $r' \in \{0, 1, 2\}$; the other part's proof is symmetric

if a_1, b_1, s_1, t_1 is a solution then $s_1, t_1, \frac{a_1}{3}, \frac{b_1}{3}$ is also a solution; further, if $s_1, t_1, \frac{a_1}{3}, \frac{b_1}{3}$ is a solution, then $\frac{a_1}{3}, \frac{b_1}{3}, \frac{s_1}{3}, \frac{t_1}{3}$ is also a solution; hence, leading to a strictly smaller (lexicographically) solution vector

- No non-trivial non-negative integral solution to $x^3 + 2y^3 = 4z^3$.

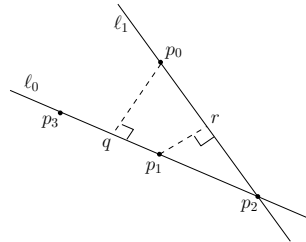
- for any solution (x_1, y_1, z_1) , considering x_1^3 must be even, x_1 must be even
 pluggin in $x_1 = 2x_2$ leads to $8x_2^3 + 2y_1^3 = 4z_1^3$, which is $4x_2^3 + y_1^3 = 2z_1^3$
 now since y_1 must be even, substituting $y_1 = 2y_2$ leads to $2x_2^3 + 4y_2^3 = z_1^3$
 again, since z_1 must be even, substituting $z_1 = 2z_2$ leads to $x_2^3 + 4y_2^3 = z_2^3$
- that is, (x_1, y_1, z_1) is a solution implies $(x_2 = \frac{x_1}{2}, y_2 = \frac{y_1}{2}, z_2 = \frac{z_1}{2})$ is a solution implies \dots implies $(x_n = \frac{x_1}{2^n}, y_n = \frac{y_1}{2^n}, z_n = \frac{z_1}{2^n})$ is a solution \dots

the infinite descent is due to $x_1 > x_2 > \dots > x_n > \dots$; the same is with the y_i s and z_i s

- *Sylvester-Gallai theorem*: If there are n ($n \geq 3$) points on the plane such that not all on a line, then there exists a line passing through exactly two of these points.

- let L be the set comprising all lines that pass through at least two points of S ; among all pairs (p, ℓ) with $p \in S$ not on $\ell \in L$, choose a pair (p_0, ℓ_0) such that p_0 has the smallest distance to ℓ_0 ; then ℓ_0 must have exactly two points

otherwise, the length of p_1r is lesser to p_0q : from similar triangles, $\frac{p_0p_2}{p_1p_2} = \frac{p_0q}{p_1r} \Rightarrow p_1r = (p_0q) \frac{p_1p_2}{p_0p_2} < p_0q$



¹a.k.a., Fermat's method of descent