(1) Fix a field $\mathcal{F}$ and choose an element $r \in \mathcal{F}$ uniformly at random from $\mathcal{F}$:

* *Verifying polynomial identities*: Given two polynomials $f(x) = \Pi_{i=1}^{n}(x - a_i)$ (in this form) and $g(x) = \sum_{i=0}^{n} c_i x^i$ (in this form), devise a Monte Carlo algorithm to verify whether $f(x) = g(x)$ in $O(n)$ expected time.

- Algorithm: Fix a field of integers $[1, j]$ and choose an element $r \in [1, j]$ uniformly at random. (The value of j is to be set shortly.) If $f(r) = g(r)$ then print "input polynomials are equal" else print "input polynomials are not equal".

  While it is immediate $f(r)$ can be computed in $O(n)$ time, by applying Horner's rule, $g(r)$ is also computable in linear time.

- This algorithm errs only if $f(x) \neq g(x)$ but $f(r) = g(r)$. Hence, it is a Monte Carlo algorithm with one-sided error. Specifically, algorithm's ouput is incorrect whenever r is a root of $f(x) - g(x) = 0$ but $f(x) \neq g(x)$. Since the number of roots of $f(x) - g(x) = 0$ is at most n, by setting $j = 1000n$, this algorithm returns a wrong answer with probability at most $\frac{n}{1000n} = \frac{1}{1000}$.

  Noting that choosing a $r \in [1, 1000n]$ takes $O(\lg n)$ time in expectation, the time complexity of this algorithm is $O(n)$ in expectation.

- Using the abundance of witnesses paradigm, choose $r_1, r_2, \ldots, r_k \in [1, 1000n]$ uniformly at random. If $f(r_i) = g(r_i)$ for every $i \in [1, k]$, then print "input polynomials are equal" else print "input polynomial are not equal". Then, the probability algorithm outputs $f(x) = g(x)$ while $f(x) \neq g(x)$ is at most $(\frac{1}{1000})^k$.

  The prob of error could also be reduced by increasing the value of j, however, the value to be used is limited by the precision available. Besides, since rand$(1, k)$ takes $O(\lg k)$ time in expectation (assuming rand$(0, 1)$ takes $O(1)$ worst-case time), the overhead due to rand$(1, j)$ needs to be accounted.

* *Verifying matrix multiplication*: Given three $n \times n$ matrices A, B, and C, devise a Monte Carlo algorithm to verify whether $AB = C$ in $O(n^2)$ time in the worst-case.

- Algorithm: Compute a vector $r = (r_1, r_2, \ldots, r_i)$ from the field $\{0, 1\}^n$, while choosing each r_i uniformly at random from $\{0, 1\}$. In $O(n^2)$ time, compute $X = A(Br)$; in $O(n^2)$ time, compute $Y = Cr$. If $X \neq Y$, then print '$AB \neq C$' else print '$AB = C$'.

- The above algorithm outputs an incorrect answer only if $AB \neq C$ but $X = Y$. ——— (1)

  That is, $AB \neq C \Leftrightarrow D = (AB - C) \neq 0 \Leftrightarrow D$ must have some nonzero entry, say d_{ij}.

  $X = Y \Leftrightarrow (AB - C)r = 0 \Rightarrow \sum_{k=1}^{n} d_{ik} r_k = 0 \Leftrightarrow r_j = \frac{-(\sum_{k=1}^{n} d_{ik} r_k) + d_{ij} r_j}{d_{ij}}$. ——— (2)

  Suppose r_j is chosen last among all r_is. By the time r_j is chosen uniformly at random from $\{0, 1\}$, equality holds whenever r_i is set to exactly one of $\{0, 1\}$. That is, (1) holds whenever r_i is chosen so that (2) is satisfied. Therefore, (1) holds with probability at most $\frac{1}{2}$.

- Algorithm utilizing abundance of witnesses paradigm: Choose k random vectors, each from $\{0,1\}^n$. If $A(Br') \neq Cr'$ for any such random vector r', then print '$AB \neq C$' else print '$AB = C$'. It is obvious that this reduces the error probability to at most $(\frac{1}{2})^k$. Obviously, this algorithm takes $O(kn^2)$ time in the worst-case.

- This technique of verifying matrix multiplication is knows as the *Freivalds' technique*.

* *Two-point sampling* for *probability amplfiication*: Consider an algorithm A that decides a language L and takes polynomial time in the worst-case, that is, for any given input string x, A determines whether $x \in L$ in polynomial time in the worst-case. Given x, A picks a random number r from $Z_p = \{0, 1, \ldots, p-1\}$ for a prime p, and computes a binary value $A(x, r)$ with the following properties: if $x \in L$ then $A(x, r) = 1$ for at least half the possible values of r and if $x \notin L$ then $A(x, r) = 0$ for all possible choices of r. That is, for any $x \in L$, for a randomly chosen r with $A(x, r) = 1$ is a *witness* that $x \in L$ with probability at least $\frac{1}{2}$, while $A(x, r) = 0$ is an evidence that $x \notin L$. On the other hand, by picking $t > 1$ values $r_1, r_2, \ldots, r_t$ independently from Z_p, and computing $A(x, r_i)$ for all of them; if for any i, $A(x, r_i) = 1$, declare that $x \in L$, else declare that $x \notin L$. Then, the probability of incorrectly classifying an input $x \in L$ is at most 2^{-t}. However, choosing t independent random numbers is expensive in that it requires $\Omega(t \lg n)$ random bits.

- In the two-point sampling (with replacement), two independent samples a and b are drawn from Z_p, needing only $2 * \lg n$ random bits. The number of random bits needed by the algorithm is called its *random bit complexity*. By using pairwise independent bits instead of mutually independent ones, random bit complexity is reduced. However, computing $A(x, a)$ and $A(x, b)$ yields an upper bound of only $\frac{1}{4}$ on the probability of incorrect classification. For the probability amplification, define $r_i = ai + b(modp)$, and compute $A(x, r_i)$ for $1 \leq i \leq t$. For any i, $A(x, r_i) = 1$, then declare that $x \in L$, else declare that $x \notin L$. Since a and b are pairwise independent, r_is are pairwise independent. Since r_is are pairwise independent, the random variables $A(x, r_i)$s are pairwise independent, for $1 \leq i \leq t$. Let $Y = \sum_{i=1}^{t} A(x, r_i)$. Assuming $x \in L$, $E[Y] \geq t/2$ and $Var[Y] = \sum_{i=1}^{t} Var[A(x, r_i)] \leq \frac{t}{4}$. The probability that all those iterations produce an incorrect classification corresponds to the event $Y = 0$, and $p(Y = 0) \leq p(|Y - E[Y]| \geq t/2) \leq \frac{Var[X]}{(t/2)^2} \leq \frac{1}{t}$. Essentially, this method helped in improving the error bound from $\frac{1}{4}$ to $\frac{1}{t}$, referred to as *probability amplification*.

(2) Fix the point of evaluation a and choose a random field over which the evaluation of a is to be performed:

* *Verifying equality of strings*: Alice has string $A : a_1, \ldots, a_n$, and Bob has string $B : b_1, \ldots, b_n$, with every $a_i, b_j \in \{0, 1\}$, Bob determines whether $A = B$ but Alice can afford to transmit only $o(n)$ bits to Bob. Devise a Monte Carlo algorithm with polynomially small error wherein Alice does transmit only $o(n)$ bits to Bob.

- Define points of evaluation $a = \sum_{i=1}^{n} a_i 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i 2^{i-1}$. Define the fingerprint function $f_p(x) = x \bmod p$ for a prime p chosen uniformly at random from $[2, \tau = n^2 \lg n]$.

  Algorithm: if $f_p(a) \neq f_p(b)$ then print $A \neq B$ else print $A = B$.

  Noting that $f_p(a) = f_p(b)$ iff p divides $|a - b|$, below, we upper bound $pr(f_p(a) = f_p(b) \mid a \neq b)$.

- Observation: Since $|a-b| < 2^n$, there can be at most n prime divisors of $|a-b|$. (Proof: If the number of distinct prime divisiors of $|a-b|$ are more than n, since each prime divisor is at least 2, the $|a-b| > 2^n$.)

  From the prime number theorem, the number of primes less than or equal to τ is asymptotically $\frac{\tau}{\ln \tau}$; again, since only n among these can be divsors of $|a-b|$, when probability is taken over the random choices of p,

  $pr(f_p(a) = f_p(b) \mid a \neq b) = pr(\text{p divides } |a-b| \mid a \neq b) \leq \frac{n}{\frac{\tau}{\ln \tau}} = \frac{n \ln \tau}{\tau} = \frac{n(\ln(n^2) + \ln(\lg(n)))}{n^2 \lg(n)} = \frac{1}{n}(\frac{2 \ln n}{\lg n} + \frac{\ln \lg n}{\lg n}) = O(\frac{1}{n})$. $\leftarrow$ error is polynomially small, desirable since the error reduces as n grows

  Further, since p is at most τ, the number of bits to be transmitted (at most p) from Alice to Bob is $O(\lg \tau)$, i.e., $O(\lg n)$.

  Note that the most expensive operation in this algorithm is computing a $p \in [2, \tau]$. However, once a p is shared between Alice and Bob, fingerprints can be taken for any A and B.

- Significantly, in this problem, we fixed the points of evaluation ($a = \sum_{i=1}^{n} a_i 2^{i-1}, b = \sum_{i=1}^{n} b_i 2^{i-1}$) and for a random prime p of a reasonably small magnitude, fingerprints were obtained by evaluating a and b over the field Z_p.

- How many numbers one needs to choose in $[1, \tau]$ with replacement before the number chosen is prime?

  View the process of randomly selecting a number and determining whether it is a prime as a Bernoulli trial. Further, we know, for a geometric random variable X that has value i if the success occurs at i^{th} trial, then $E[X] = \frac{1}{p}$. From the prime number theorem, $p = \frac{(\tau / \ln(\tau))}{\tau} = \frac{1}{\ln(\tau)}$. Hence, the expected number of trials needed to obtain a prime number in $[1, \tau]$ is $\ln \tau$. Then, with AKS algorithm, the average-time to find a prime in $[1, \tau]$ is $O((\lg \tau)^7)$.

* *Pattern matching*: Let the alphabet Σ be $\{0, 1\}$. Given a text string $T \in \Sigma^*$ of length n and a pattern string $P \in \Sigma^*$ of length m, for $m < n$, devise a Monte Carlo algorithm to find the smallest value of shift s such that $T[s+1 \ldots s+m] = P$.

- Algorithm: Choose a prime number p in $f_p(x) = x \bmod p$ unformly at random from the set of numbers in the range $[1, \tau = n^2 m \lg(n^2 m)]$. If $f_q(T[s+1 \ldots s+m]) = f_q(P)$ then output shift s and exit; otherwise, try with shift $s+1$.

- Since $|T[s+1 \ldots s+m] - P|$ is an m-bit positive integer with value $< 2^m$, there can be at most m distinct prime divisors to $|T[s+1 \ldots s+m] - P|$. The number of primes smaller than τ is asymptotically $\frac{\tau}{\ln \tau}$; again, only m among these can be divsors of $|T[s+1 \ldots s+m] - P|$.

  When probability is taken over the random choices of q,

  $pr((f_q(T[s+1 \ldots s+m]) = f_q(P)) \mid (T[s+1 \ldots s+m] \neq P))$

  $= pr(q \text{ divides } (|T[s+1 \ldots s+m] - P|) \mid (T[s+1 \ldots s+m] \neq P))$

  $\leq \frac{m}{\frac{\tau}{\ln \tau}} = O(\frac{m \lg \tau}{\tau}) = O(\frac{1}{n})$.

- The worst-case time, ignoring the number of times to iterate for finding a prime, is $O(n + m)$.

(3) Interpret the bit vectors a and b as the n-bit integers a and b; fix a prime number $p > 2^n$; choose a random polynomial over the field Z_p, and obtain the fingerprints by evaluating this polynomial at the integers a and b, performing all arithmatic over the field Z_p, and then reducing the resulting values modulo a number of magnitude close to $\lg n$.

- Let U be the universe comprising keys and let T be a hash table. Also, let $|T| = m$ and $|U| > m$.

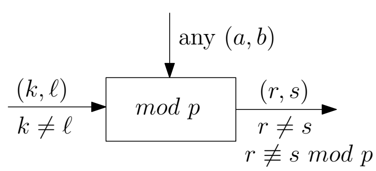
  A collection of hash functions $\mathcal{H}$ is called a 2-*universal hash family* whenever (i) each function in $\mathcal{H}$ is from U to T, and (ii) for any pair of distinct keys $k_i, k_j \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(k_i) = h(k_j)$ is at most $\frac{|\mathcal{H}|}{m}$.

  A hash function h is a 2-*universal hash function* if h is chosen, uniformly at random and independent of keys being stored in T, from a 2-universal hash family. (Choosing independent of keys being stored ensures lesser number of collisions in expectation, even if keys are chosen by an adversary.)

  With any 2-universal hash function h, for any pair of distinct keys $k_i, k_j \in U$, $pr(h(k_i) = h(k_j)) \leq \frac{|\mathcal{H}|/m}{|\mathcal{H}|} = \frac{1}{m}$. That is, a 2-universal hash function obeys simple uniform hashing.

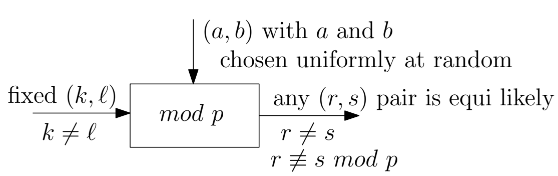
  Below, we assume $a \in [1, p-1], b \in [0, p-1]$, $U \in [0, p-1]$, and obviously every key $k \in U$. Let k, ℓ be two distinct keys, we define, $r = (ak + b) \bmod p$ and $s = (a\ell + b) \bmod p$. The following proofs are reproduced from pages 267-268 of [CLRS].

- Lemma 1: For (k, ℓ) with $k \neq \ell$, any fixed (a, b), leads to (r, s) with $r \neq s$ and $r \not\equiv s \bmod p$.



  We know, $r - s \equiv a(k - \ell) \bmod p$. Since $a \neq 0$ (from the definition of $\mathcal{H}_{pm}$), since $a \not\equiv 0 \bmod p$ (as $a \in [1, p-1]$), since $k - \ell \not\equiv 0 \bmod p$ (as every key $k \in [0, p-1]$), and since their product must also be nonzero modulo p, $r - s \not\equiv 0 \bmod p$. Since p is a prime, this implies, $r \not\equiv s \bmod p$.

- Lemma 2: For (k, ℓ) with $k \neq \ell$, if (a, b) is chosen uniformly at random from $[1, p-1] \times [0, p-1]$, then the resulting pair (r, s) is equally likely to be any pair of distinct values modulo p.



Since $r = (ak + b) \bmod p$ and $s = (a\ell + b) \bmod p$,
$a = (r - s)((k - \ell)^{-1} \bmod p) \bmod p$, and
$b = (r - ak) \bmod p$.

From these, we can find a and b, given r and s. Hence, each of the possible $p(p-1)$ choices for the pair (a, b) yields a different resulting pair (r, s) with $r \neq s$.

Since there are only $p(p-1)$ possible pairs (r, s) with $r \neq s$ (from Lemma 1), there is a one-to-one correspondance between pairs (a, b) and pairs (r, s) with $r \not\equiv s \bmod p$.

Thus, for any given pair of inputs k and ℓ, if we pick (a, b) uniformly at random from $[1, p-1] \times [0, p-1]$, the pair (r, s) is equally likely to be any pair with $r \not\equiv s \bmod p$.

Choosing tuple (a, b) uniformly at random from any of $p(p-1)$ number of tuples,
$pr(r \neq s) = \frac{1}{p(p-1)}$. ——— (A)

- Theorem: Assuming every key $k \in [0, p-1]$,
  $\mathcal{H}_{pm} = \{h_{ab}(k) = ((ak + b) \bmod p) \bmod m : a \in [1, p-1], b \in [0, p-1]$, for a prime $p > m\}$
  is a 2-universal hash family.

[Here, choosing $h \in \mathcal{H}_{pm}$ uniformly at random is equivalent to choosing a and b uniformly at random respectively from $[1, p-1]$ and $[0, p-1]$. Significantly, given the universe and the hash table are fixed, to save any function from $\mathcal{H}_{pm}$, one needs to store only a and b.]

$$\begin{array}{c} (r,s) \\ \longrightarrow \\ r \neq s \\ r \not\equiv s \text{ mod } p \end{array} \boxed{\textit{mod } m} \begin{array}{c} (r' = r \bmod m, s' = s \bmod m) \\ \longrightarrow \\ \text{with } pr(r' \equiv s' \bmod m) \leq \frac{1}{m} \end{array}$$

For $k \neq \ell$, it is immediate that $pr(h_{ab}(k) = h_{ab}(\ell)) = pr(r \equiv s \bmod m)$.

From (A), we know $pr(r \neq s)$ is $\frac{1}{p(p-1)}$; leading to,
$pr(h_{ab}(k) = h_{ab}(\ell)) = \frac{1}{p(p-1)} \cdot |\{(r, s) : r \neq s$ and $r \equiv s \mod m\}|$.

For a given value of r, of the $p-1$ possible remaining values for s, the number of values s such that $s \neq r$ and $s \equiv r \bmod m$ is at most $\lceil \frac{p}{m} \rceil - 1 \leq (\frac{p+m-1}{m}) - 1 = \frac{p-1}{m}$.

Since r can assume p number of values, $pr(h_{ab}(k) = h_{ab}(\ell)) \leq \frac{1}{p(p-1)} \cdot \frac{p(p-1)}{m} = \frac{1}{m}$.

References:

- Randomized Algorithms by R. Motawani and P. Raghavan. [Sections 3.4, 7.1-7.2, and 7.4-7.6.]
- Introduction to Algorithms by T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein, Third Edition. [pg 267-268]