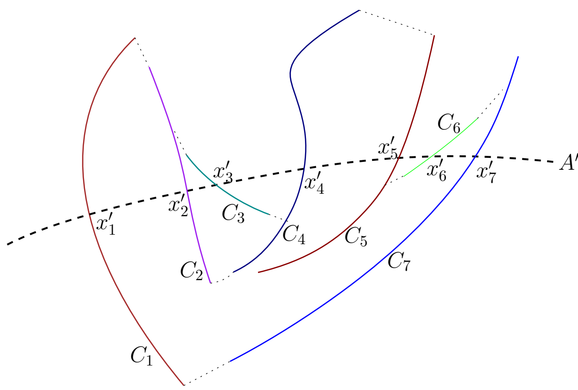
- A subset of a poset such that every two elements of this subset are comparable is called a *chain*. A *maximal chain* is a chain that is not a proper subset of any other chain. A *maximum chain* is a chain that has cardinality at least as large as every other chain. The *height of a poset* is the cardinality of a maximum chain.

- A subset of a poset is called an *antichain* if every two elements of this subset are not comparable. A *maximal antichain* is an antichain that is not a proper subset of any other antichain. A *maximum antichain* is an antichain that has cardinality at least as large as every other antichain. The *width of a poset* is the cardinality of a maximum antichain.

- *Dilworth's theorem*: If w is the width of a finite non-empty poset $(S, \preccurlyeq)$, then there is a partition of elements in S into w chains.

  Proof is by induction on the cardinality of S:

  * Basis: Consider a set S with one element, say $S = \{x\}$. In $(S, \preccurlyeq)$, the only maximum antichain is $\{x\}$, its size is 1, and $C_1 = \{x\}$ with $C_1 = S$.
  * Induction hypothesis: Let x be a maximal element of $(S, \preccurlyeq)$. And, let S' be $S - \{x\}$. For every set $S'' \subseteq S'$, if w'' is the width of $(S'', \preccurlyeq)$, then there is a partition of elements in S'' into w'' chains.

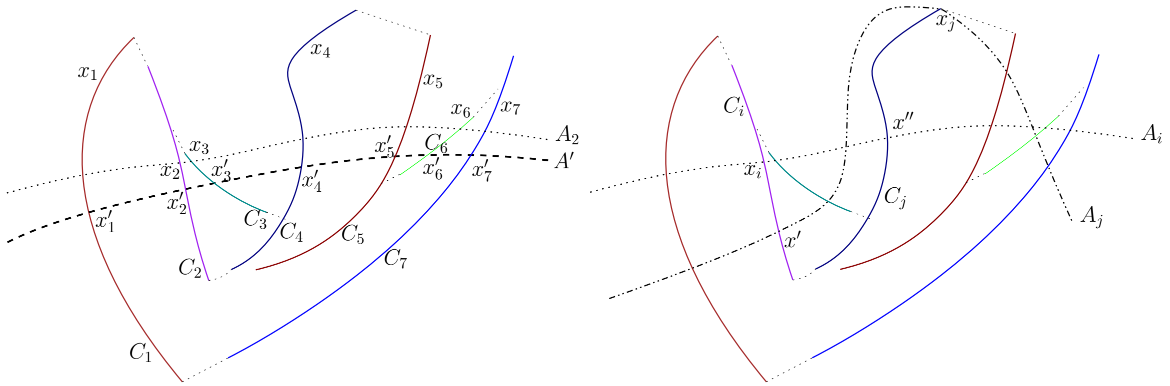


Illustrating a maximum antichain A' comprising elements $x'_1, x'_2, \ldots, x'_7$ with a black dashed line. Also, shown a partitioning of $S' = S - \{x\}$ into seven chains ($\mathcal{C}' = \{C_1, C_2, \ldots C_7\}$), each is in a different color.

Let A' be a maximum antichain of poset $(S', \preccurlyeq)$ comprising w' elements. Also, let $\mathcal{C}' = \{C_1, C_2, \ldots, C_{w'}\}$ be the chain partitioning of elements in S' due to induction hypothesis.

  - Lemma 1: For every $i \in [1, w']$, $|C_i \cap A'| = 1$.
    Proof: It is obvious that $|C_i \cap A'| \leq 1$. Suppose $C_j \cap A' = \phi$ for $C_j \in \mathcal{C}'$. This implies, from the pigeonhole principle, there exists a chain $C_{j'} \in \mathcal{C}'$ such that $C_{j'}$ contributes at least two elements, say y' and y'', to A' given $|A'| = w'$. However, then since y' and y'' are comparable, A' is not an antichain. $\square$

  - Lemma 2: For $i = 1, 2, \ldots, w'$, let x_i be the maximal element in C_i that belongs to a maximum antichain A_i of $(S', \preccurlyeq)$. Then, $A = \{x_1, x_2, \ldots, x_{w'}\}$ is an antichain of $(S', \preccurlyeq)$.
    Proof: For every i, A_i always exists, since an element a'_i of C_i belongs to antichain A'. (For example, a_i can be found by walking along C_i from top to bottom.) Refer to left figure.

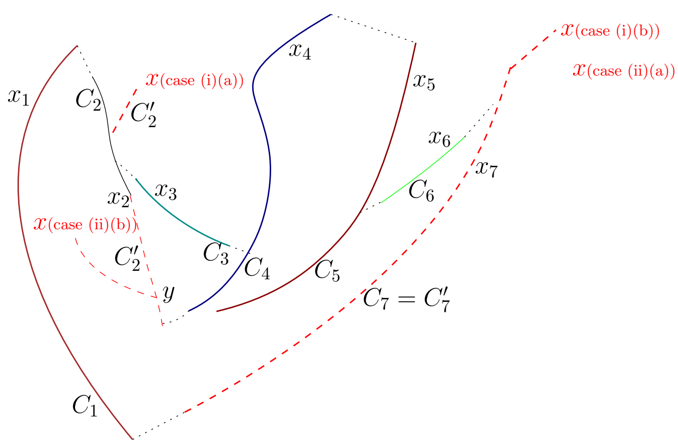


On the left, illustrating x_i is the first element while walking along chain C_i from top to bottom for every $i \in [1, 7]$, so that there is a maximum antichain A_i that contains x_i. For example, maximum antichain A_2 has x_2 whereas no element above x_2 along C_2 can belong to a maximum antichain. Noting that, for every $i \in [1, 7]$, x'_i of A' is $\preccurlyeq x_i$, it is guaranteed for x_i to exist on C_i.
On the right figure, illustrating maximum antichains A_i and A_j, respectively containing x_i and x_j.

Suppose $x_i \preccurlyeq x_j$. Let $x' \in C_i \cap A_j$. Then, since $x' \preccurlyeq x_i$, from transitivity, $x' \preccurlyeq x_j$, leading to a contadiction of A_j being an antichain. Analogously, suppose $x_j \preccurlyeq x_i$. Let $x'' \in C_j \cap A_i$. Then, since $x'' \preccurlyeq x_j$, from transitivity, $x'' \preccurlyeq x_i$, leading to a contadiction of A_i being an antichain. Refer to right figure.

For every $x_i, x_j \in A$, since $x_j \not\preccurlyeq x_i$ and $x_i \not\preccurlyeq x_j$, A is an antichain. □

* Induction step: Consider the poset $(S, \preccurlyeq)$. Since x is a maximal element of this poset, there are two possibilities: (i) $x_i \preccurlyeq x$ for some $i \in [1, \ldots, w']$, or (ii) $x_i \not\preccurlyeq x$ for every $i \in [1, \ldots, w']$. Note that w' is the width of $(S', \preccurlyeq)$ and S' is partitioned into w' chains.



Illustrating Case (i)(a) wherein x is above x_2, Case (i)(b) wherein x is above x_7. In both the Case (i)(a) and Case (i)(b), elements along the red colored dashed line are removed before applying induction hypothesis to the rest of the poset; significantly, due to the choice of x_2, no element in $C_2 - C'_2$ belongs to a maximum antichain. In Case (ii)(a), x is not related to any of the elements of S'. In Case (ii)(b), there is an element y in S that precedes x but that y is below x_2. That is, both in Case (ii)(a) and in Case (ii)(b), x is not related to any x_i for $i \in [1, \ldots, 7]$.

In Case (i), consider $(S' - C'_i, \preccurlyeq)$, where C'_i is the chain underneath x_i (including x_i) in $(S', \preccurlyeq)$. Since no element in $C_i - C'_i$ belongs to a maximum antichain, the width of $(S' - C'_i, \preccurlyeq)$ is $w' - 1$. Since $S' - C'_i \subseteq S'$, from the induction hypothesis (by the means of strong induction), $S' - C'_i$ is partitioned into $w' - 1$ chains. By including x into S', S can be partitioned into w' chains, consisting these $w' - 1$ chains together with C'_i, and the width of the resulting poset is w'. (That is, the width of $(S, \preccurlyeq)$ does not change by including x into S'.) Refer to above figure.

In Case (ii), applying induction hypothesis to $(S', \preccurlyeq)$ leads to partitioning S' into w' chains $C_1, C_2, \ldots$, $C_{w'}$. These with a maximal chain comprising x is a partition of S into $w' + 1$ chains. Further, due to Lemma 2 and since no x_i is related to x, $A' \cup \{x\}$ is a maximum antichain of size $w' + 1$. Refer to above figure. $\square$

- *Sperner's theorem*: The size of a largest antichain of any poset $(\mathcal{P}(S), \subseteq)$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, where $S = \{1, 2, \ldots, n\}$.

For any fixed k and for any two k-sets S', S'' belonging to $\mathcal{P}(S)$, neither $S' \subseteq S''$ nor $S'' \subseteq S'$. Hence, for any fixed k, all the k-sets of $\mathcal{P}(S)$ together form an antichain. Fixing k to $\lfloor \frac{n}{2} \rfloor$, yields an antichain whose size is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Each permutation π of elements in S yields a speicific chain, $\emptyset \subseteq \{\pi(1)\} \subseteq \{\pi(1), \pi(2)\} \subseteq \ldots \subseteq \{\pi(1), \ldots, \pi(n)\}$. Therefore, there are $n!$ chains in $(\mathcal{P}(S), \subseteq)$.

Each permutation π' of elements in $A \subseteq S$ yields a specific chain, $\emptyset \subseteq \{\pi'(1)\} \subseteq \{\pi'(1), \pi'(2)\} \subseteq \ldots \subseteq \{\pi'(1), \ldots, \pi'(|A|)\}$. Analogously, each permutation of elements in $S - A$ yeilds a specific chain. The concatentation of any two chains, by including a link between the maximum element of the former type and the minimum element of the latter type, is a maximum chain that contains A. Considering the number of permutations of elements in A and in $S - A$, there are $|A|!(n - |A|)!$ chains that contain A. That is, for any A with $|A| = k$, beloning to any antichain $\mathcal{A}$, there are $k!(n - k)!$ chains that contain A.

Since no two elements of $\mathcal{A}$ belong to any chain, the number of chains containing all the elements of $\mathcal{A}$ together is $\sum_{k=0}^{n} n_k k!(n-k)!$, where n_k is the number of k-sets in $\mathcal{A}$. Since the total number of chains is $n!$, $\sum_{k=0}^{n} n_k k!(n-k)! \leq n! \Rightarrow \sum_{k=0}^{n} \frac{n_k}{\binom{n}{k}} \leq 1 \Rightarrow \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} n_k \leq 1$. Noting $\sum_{k=0}^{n} n_k$ is the total number of elements of $\mathcal{A}$ (as it considers all k-sets in $\mathcal{A}$ for every k), $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

References:

- Proofs from the BOOK by M. Aigner and G. M. Ziegler.