- The problem of finding a MIS in undirected graphs has no polynomial time $\frac{1}{n^{1-\epsilon}}$-approximation algorithm for any fixed $\epsilon > 0$ unless $P = NP$. ← not proved in class

- For any input graph $G(V, E)$, consider the following greedy algorithm to compute an approximate MIS of G: Repeatedly remove a node v of minimum degree and every neighbor u of v from the current graph, and place v in S.

\- It is easy to see that S is an independent set of G.

\- And, $|S| \geq \frac{n}{\Delta+1}$: Charging every node $u \notin S$ to v where u and v are as in the above algorithm, each node in S is charged at most Δ times; and the bound is due to $|S| + |V \backslash S| = n$.

\- Further, $|S| \geq \frac{1}{\Delta}|OPT|$: Charging every node u in $OPT \backslash S$ to a node v in $S \backslash OPT$, noting u is a neighbor of v, the number of nodes charged to a node $v \in S \backslash OPT$ is at most Δ.

- There is a polynomial time 6-approximation algorithm for the maximum independent set problem in planar graphs.

\- Iterative algorithm: Repeatedly remove a node v of degree at most five and its neighbors from the current graph; place v into the set being computed. This yields a 6-approximation.

\- Divide and conquer algorithm: Noting planar graph separator theorem yields three sets A, B, and S with S being the separator, remove all the nodes in S, recursively compute an independent sets for planar graphs induced by A and B and return their union. When the number of vertices drops below t, find the independent set by brute-force, where t is a parameter to be fixed later.

Settng up a recurrence (for $n \geq t$, $E(n) = O(\sqrt{n}) + E(\beta n) + E((1-\beta)n)$ where $1/3 \leq \beta \leq 2/3$), the number of nodes removed over all the recursive calls is $O(\frac{n}{\sqrt{t}})$. By setting $t = \Theta(1/\epsilon^2)$, the number of nodes thrown away is $O(n\epsilon)$. Hence, this algorithm is indeed a PTAS.

- Given a set $\mathcal{D}$ of n pairwise disjoint disks in $\mathbb{R}^2$, there is a square T such that (i) at most $O(\sqrt{n})$ disks in $\mathcal{D}$ intersect the $bd(T)$, (ii) at most $\frac{4}{5}n$ disks of $\mathcal{D}$ lie completely in the interior of T, and (iii) at most $\frac{4}{5}n$ disks of $\mathcal{D}$ lie completely in the exterior of T.

Proof: Let T_0 be the smallest radius square that contains at least $\frac{n}{5}$ centers of the disks in $\mathcal{D}$. Assume $rad(T_0) = 1$. Consider the family $\mathcal{F} = \{T \,|\, 1 \leq rad(T) \leq 2, center(T) = center(T_0)\}$ of squares. We note each $T \in \mathcal{F}$ can be covered by four translates of T_0. Considering the minimality of T_0, each such translate contains at most $\frac{n}{5}$ centers. Thus, T can contain at most $\frac{4n}{5}$ centers in its iterior, implying at least $\frac{n}{5}$ centers lie in its exterior.
Let $\mathcal{D}_1 = \{D \in \mathcal{D} \,|\, rad(D) \leq \frac{1}{\sqrt{n}}\}$ and $\mathcal{D}_2 = \{D \in \mathcal{D} \,|\, rad(D) > \frac{1}{\sqrt{n}}\}$. As each $T \in \mathcal{F}$ has radius at most 2, $bd(T)$ intersects $O(\sqrt{n})$ disks in $\mathcal{D}_2$. And the expected number of disks in $\mathcal{D}_1$ intersected by the boundary of a square T picked from $\mathcal{F}$ uniformly at random is $\sum_{D \in \mathcal{D}_1} pr(bd(T) \text{ intersects } D) \leq \sum_{D \in \mathcal{D}_1} (2 \cdot rad(D)) \leq \sum_{D \in \mathcal{D}_1} \frac{2}{\sqrt{n}} = O(\sqrt{n})$.

\- Recursively applying the above theorem yields a PTAS for computing the maximum independent set of pairwise disjoint disks in the plane.