MA 102 (Multivariable Calculus)

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Partial and directional derivatives, differentiability

(1) The *kinetic energy* of an object with a constant mass m and position $\mathbf{r}(t) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$ is defined to be $K(t) := \frac{1}{2}mv^2(t)$, where $v(t) := \|\mathbf{r}'(t)\|$. Determine $K'(t)$.

(2) Find the unit tangent vector to $\mathbf{r}(t) = (e^t, 2t, 2e^{-t})$. Also find the speed of a moving object with position $\mathbf{r}(t) = (3\sin(2t), 5\cos(2t), 4\sin(2t))$ in feet at time $t \in \mathbb{R}$ in seconds.

(3) The *angular momentum* of a particle with position $\mathbf{r}(t) \in \mathbb{R}^3$ at time $t \in \mathbb{R}$ and a constant mass m is $\mathbf{L}(t) := m(\mathbf{r}(t) \times \mathbf{v}(t))$, where $\mathbf{v}(t)$ is the velocity and $\mathbf{r} \times \mathbf{v}$ is the *cross product* of $\mathbf{r}$ and $\mathbf{v}$. Show that $\mathbf{L}'(t) = m(\mathbf{r}(t) \times \mathbf{a}(t))$, where $\mathbf{a}(t)$ is the acceleration.

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0$ and $f(x,y) = \dfrac{xy}{x^2+y^2}$. Show that f is not continuous at $(0,0)$ but the partial derivatives of f exist on $\mathbb{R}^2$. Show that the partial derivatives are not continuous at $(0,0)$.

(5) Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, where U is open. If the first order partial derivatives of f exist on U and are bounded then show that f is continuous on U.

(6) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \qquad \text{for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$ and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

(7) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f_x(x,y) = 0 = f_y(x,y)$ for all $(x,y) \in \mathbb{R}^2$ then show that f is a constant function.

(8) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(0,0) = 0 = g(0,0)$ and, for $(x,y) \neq (0,0)$,

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}, \qquad g(x,y) = \frac{\sin^2(x+y)}{|x|+|y|}.$$

Examine differentiability and the existence of partial and directional derivatives of f and g at $(0,0)$.

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 0$ if $y = 0$ and and $f(x,y) = \dfrac{y}{|y|}\sqrt{x^2+y^2}$, if $y \neq 0$. Show that f is continuous at $(0,0)$, $D_u f(0,0)$ exists for all unit vector u but f is not differentiable at $(0,0)$.

(10) Find the directional derivative of $f(x,y) = y^3 - 2x^2 + 3$ at the point $(1,2)$ in the direction of $u := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Also, find the directional derivative of $f(x,y) = \log(x^2+y^2)$ at $(1,-3)$ in the direction of $u := (2,-3)$.

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(11) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0,0)$. Suppose that for $u := (3/5, 4/5)$ and $v := (1/\sqrt{2}, 1/\sqrt{2})$, we have $D_u f(0,0) = 12$ and $D_v f(0,0) = -4\sqrt{2}$. Then determine $f_x(0,0)$ and $f_y(0,0)$.

(12) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that $\partial_i f(x,y)$ exists and g is differentiable at $f(x,y)$. Show that $\partial_i (g \circ f)(x,y)$ exists and $\partial_i (g \circ f)(x,y) = g'(f(a))\partial_i f(x,y)$.

(13) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Using chain rule determine the partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$(i) f(x,y) := g(xy^2 + 1), \quad (ii) f(x,y) := g(4x + 7y), \quad (iii) f(x,y) := g(x - y).$$

Also, examine differentiability of f and determine the derivative, if it exists.

(14) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}^2$ and suppose that $\nabla f(a) \neq 0$. Show that the maximum value of the directional derivative $D_u f(a)$ is $\|\nabla f(a)\|$ and is attained in the direction of $\nabla f(a)$ with $u = \nabla f(a)/\|\nabla f(a)\|$. Also show that the minimum value of $D_u f(a)$ is $-\|\nabla f(a)\|$ and is attained in the direction of $-\nabla f(a)$.

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