MA 102 (Multivariable Calculus)

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Continuity and limits of functions on $\mathbb{R}^n$

(1) Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,

(a) $f(x,y) := \begin{cases} xy\cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ (b) $f(x,y) := \begin{cases} 1 & \text{if } x > 0 \;\&\; 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

(c) $f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ (d) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(e) $f(x,y) := \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ (f) $f(x,y) := \begin{cases} \frac{x^3y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Define $\phi_y : \mathbb{R} \to \mathbb{R}$ by $\phi_y(t) := f(t,y)$ for each fixed y, and $\psi_x : \mathbb{R} \to \mathbb{R}$ by $\psi_x(s) := f(x,s)$ for each fixed x. Prove or disprove: f is continuous if and only if ϕ_y and ψ_x are continuous for each $(x,y) \in \mathbb{R}^2$.

(3) Let $A \subset \mathbb{R}^n$ be nonempty such that every continuous function $f : A \to \mathbb{R}$ attains its maximum and minumum. Show that A is compact.

(4) Suppose that f is uniformly continuous on $A \subset \mathbb{R}^n$. If $(\mathbf{x}_k)$ is a Cauchy sequence in A, then show that $(f(\mathbf{x}_k))$ is a Cauchy sequence. Show by an example that if f is continuous on A then $(f(\mathbf{x}_k))$ may not be a Cauchy sequence.

(5) Examine whether the following limits exist and find their values if they exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|}{y^2}e^{-|x|/y^2}$ (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ (f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$

(6) For the functions $f : \mathbb{R}^2 \to \mathbb{R}$ given below examine continuity at $(0,0)$ and show that **exactly two** of the following limits exist and are equal:

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{x\to 0}\lim_{y\to 0} f(x,y), \qquad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

(a) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(b) $f(x,y) := \begin{cases} y + x\sin(1/y) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$ (c) $f(x,y) := \begin{cases} x + y\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

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(7) For the functions $f : \mathbb{R}^2 \to \mathbb{R}$ given below show that **exactly one** of the following limits exists:

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{x\to 0}\lim_{y\to 0} f(x,y), \qquad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

(a) $f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$

(b) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} + x\sin(1/y) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

(c) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} + y\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(8) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) := \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ Show that the iterated limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exist and are unequal.

Moral: *Existence of limit does not guarantee existence of iterated limits and vice-versa. Iterated limits when exist may be unequal. However, if limit and iterated limits exist then they are all equal.*

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