MA 102 (Multivariable Calculus)

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The Euclidean space $\mathbb{R}^n$, Norms, convergence of sequences in $\mathbb{R}^n$,

(1) Consider the Euclidean norm $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and show that $|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,| \leq \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if
$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

(2) Let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be such that (a) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = 0$, (b) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, (c) $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, and (d) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. The function $\langle \cdot, \cdot \rangle$ is called an innerproduct on $\mathbb{R}^n$. Define $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for $\mathbf{x} \in \mathbb{R}$. Show that $p(t) := \|\mathbf{x} + t\mathbf{y}\|^2 \geq 0$ for all $t \in \mathbb{R}$ and hence prove the Cauchy-Schwarz inequality $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that the equality holds if and only if $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \in \mathbb{R}$. Show also that $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

(3) Show that $\|\mathbf{x}\|_1 := |x_1| + \cdots + |x_n|$, $\|\mathbf{x}\| := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$, and $\|\mathbf{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$ are norms on $\mathbb{R}^n$. Let $(\mathbf{x}_k)$ be a sequence in $\mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Show that $(\mathbf{x}_k)$ converges to $\mathbf{x}$ w.r.t. $\|\cdot\|_1 \iff (\mathbf{x}_k)$ converges to $\mathbf{x}$ w.r.t. $\|\cdot\| \iff (\mathbf{x}_k)$ converges to $\mathbf{x}$ w.r.t. $\|\cdot\|_\infty$.

(4) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Show that $\mathbf{x}_k \to \mathbf{x}$ in $\mathbb{R}^n$ if and only if for every $\mathbf{y} \in \mathbb{R}^n$ the sequence $(\langle \mathbf{x}_k, \mathbf{y} \rangle) \subset \mathbb{R}$ converges to $\langle \mathbf{x}, \mathbf{y} \rangle$, that is, $\langle \mathbf{x}_k, \mathbf{y} \rangle \to \langle \mathbf{x}, \mathbf{y} \rangle$ in $\mathbb{R}$.

(5) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ be such that $\mathbf{x}_k \to \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Show that the sequence $(\|\mathbf{x}_k\|) \subset \mathbb{R}$ converges to $\|\mathbf{x}\|$. Additionally suppose that $\mathbf{x} \neq 0$ and $\mathbf{x}_k \neq 0$ for all k, and define $\mathbf{y}_k := \mathbf{x}_k / \|\mathbf{x}_k\|$ and $\mathbf{y} := \mathbf{x}/\|\mathbf{x}\|$. Show that $\mathbf{y}_k \to \mathbf{y}$.

(6) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Suppose that $\mathbf{x}_k \to \mathbf{x}$ and that $\langle \mathbf{x}_k, \mathbf{y} \rangle = 0$ for all k. Show that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

(7) Let $\alpha \in (0, 1)$ and let $\mathbf{x}_n = (n^3\alpha^n, \frac{1}{n}[n\alpha])$ for all $n \in \mathbb{N}$. (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.) Examine whether the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$. Also, find $\lim\limits_{n\to\infty} \mathbf{x}_n$ if it exists.

(8) (**Bolzano-Weierstrass Theorem**) Let $(\mathbf{x}_n) \subset \mathbb{R}^2$ be bounded. Show that $(\mathbf{x}_n)$ has a subsequence that converges in $\mathbb{R}^2$.

(9) (**Heine-Borel Theorem**) Let $S \subset \mathbb{R}^n$. Show that S is compact if and only if S is bounded and closed in $\mathbb{R}^n$.

(10) Let $(\mathbf{x}_k) \subset \mathbb{R}^n$ be such that $\mathbf{x}_k \to \mathbf{x}$ and that $\|\mathbf{x}\| = r > 0$. Show that there exists $p \in \mathbb{N}$ such that $\|\mathbf{x}_k\| > r/2$ for all $k \geq p$.

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