Lecture 9: Implicit function theorem, constrained extrema and Lagrange multipliers

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What does the Implicit function theorem say?

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be C^1. Consider the curve

$$\mathrm{V}(F) := \{(x, y) \in \mathbb{R}^2 : F(x, y) = 0\}.$$

Does there exist $f : \mathbb{R} \to \mathbb{R}$ such that $V(F) = \mathrm{Graph}(f)$?
Equivalently, can $F(x, y) = 0$ be solved either for x or for y?

Considering $F(x, y) := x^2 + y^2 - 1$, it follows that $F(x, y) = 0$ cannot be solved for y or x.

The implicit function theorem says that if $F(a, b) = 0$ and $\nabla F(a, b) \neq (0, 0)$ then in a neighbourhood of (a, b), we have

$$\mathrm{V}(F) = \mathrm{Graph}(f)$$

for some function f.

Implicit function theorem

Theorem: Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1, where U is open. Consider the curve $\mathrm{V}(F) := \{(x, y) \in U : F(x, y) = 0\}$. Let $(a, b) \in \mathrm{V}(F)$. Suppose that $\partial_y F(a, b) \neq 0$.

- Then there exists $r > 0$ and a C^1 function $g : (a - r, a + r) \to \mathbb{R}$ such that $F(x, g(x)) = 0$ for $x \in (a - r, a + r)$.
- For $W := (a - r, a + r) \times (b - r, b + r)$, we have

$$W \cap \mathrm{V}(F) = \mathrm{Graph}(g).$$

- Further, $\partial_x F(a, b) + \partial_y F(a, b) g'(a) = 0$.

Implicit derivative

Thus if $\partial_y f(a, b) \neq 0$ then in some disk about (a, b) the set of points (x, y) satisfying $F(x, y) = 0$ is the graph of a function $y = g(x)$ with

$$\frac{\mathrm{d}y}{\mathrm{d}x}|_{x=a} = g'(a) = -F_x(a, b)/F_y(a, b).$$

Example: Consider $F(x, y) := e^{x-2+(y-1)^2} - 1$ and the equation $F(x, y) = 0$. Then $F(2, 1) = 0$, $\partial_x F(2, 1) = 1$, and $\partial_y F(2, 1) = 0$.

Hence $x = g(y)$ for some C^1 function $g : (2 - r, 2 + r) \to \mathbb{R}$. Moreover, $g'(1) = 0$.

Implicit function theorem for $F : \mathbb{R}^n \to \mathbb{R}$

Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be C^1, where U is open. Consider the curve $\mathrm{V}(F) := \{(\mathbf{x} \in U : F(\mathbf{x}, y) = 0\}$. Let $(\mathbf{a}, b) \in \mathrm{V}(F)$. Suppose that $\partial_y F(\mathbf{a}, b) \neq 0$.

- Then there exists $r > 0$ and a C^1 function $g : B(\mathbf{a}, r) \to \mathbb{R}$ such that $F(\mathbf{x}, g(\mathbf{x})) = 0$ for $\mathbf{x} \in B(a, r)$.
- For $W := B(\mathbf{a}, r) \times (b - r, b + r)$, we have
$$W \cap \mathrm{V}(F) = \mathrm{Graph}(g).$$
- Further, $\partial_i F(\mathbf{a}, b) + \partial_y F(\mathbf{a}, b)\partial_i g(\mathbf{a}) = 0, i = 1, 2, \ldots, n$

Constrained extrema of $f : \mathbb{R}^n \to \mathbb{R}$

Let $U \subset \mathbb{R}^n$ be open and $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then

$$\text{Maximize or Minimize } f(\mathbf{x})$$
$$\text{Subject to the constraint } g(\mathbf{x}) = \alpha.$$

Example: Find the extreme values of $f(x, y) = x^2 - y^2$ along the circle $x^2 + y^2 = 1$.

It turns out that f attains maximum at $(0, \pm 1)$ and minimum at $(\pm 1, 0)$ although $\nabla f(0, \pm 1) \neq 0$ and $\nabla f(\pm 1, 0) \neq 0$.

Test for constrained extrema of $f : \mathbb{R}^2 \to \mathbb{R}$

Theorem: Let $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be C^1. Suppose that f has an extremum at $(a, b) \in U$ such that $g(a, b) = \alpha$ and that $\nabla g(a, b) \neq (0, 0)$. Then there is a $\lambda \in \mathbb{R}$, called Lagrange multiplier, such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Proof: Let $\mathbf{r}(t)$ be a local parametrization of the curve $g(x, y) = \alpha$ such that $\mathbf{r}(0) = (a, b)$. Then $f(\mathbf{r}(t))$ has an extremum at $t = 0$. Therefore

$$\frac{\mathrm{d}f(\mathbf{r}(t))}{\mathrm{d}t}\Big|_{t=0} = \nabla f(a, b) \bullet \mathbf{r}'(0) = 0.$$

Now $g(\mathbf{r}(t)) = \alpha \Rightarrow \nabla g(a, b) \bullet \mathbf{r}'(0) = 0$. This shows that $\mathbf{r}'(0) \perp \nabla g(a, b)$ and $\mathbf{r}'(0) \perp \nabla f(a, b)$. Hence $\nabla f(a, b) = \lambda \nabla g(a, b)$ for some $\lambda \in \mathbb{R}$. ■

Method of Lagrange multipliers for $f : \mathbb{R}^2 \to \mathbb{R}$

To find extremum of f subject to the constraint $g(x, y) = \alpha$, define $L(x, y, \lambda) := f(x, y) - \lambda(g(x, y) - \alpha)$ and solve the equations

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \Rightarrow \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \\
\frac{\partial L}{\partial y} &= 0 \Rightarrow \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}, \\
\frac{\partial L}{\partial \lambda} &= 0 \Rightarrow g(x, y) = \alpha.
\end{aligned}$$

• The auxiliary function $L(x, y, \lambda)$ is called Lagrangian which converts constrained extrema to unconstrained extrema.

• Critical points of L are eligible solutions for constrained extrema.

Example

Find the extreme values of $f(x, y) = x^2 - y^2$ along the circle $x^2 + y^2 = 1$.

The equations $f_x = \lambda g_x, f_y = \lambda g_y$ and $g(x, y) = 1$ give $2x = \lambda 2x, -2y = \lambda 2y$ and $x^2 + y^2 = 1$. The first equation shows either $x = 0$ or $\lambda = 1$.

If $x = 0$ then $y = \pm 1 \Rightarrow \lambda = -1$. Thus $(x, y, \lambda) := (0, \pm 1, -1)$ are eligible solutions.

If $\lambda = 1$ then $y = 0 \Rightarrow x = \pm 1$. Thus $(x, y, \lambda) := (\pm 1, 0, 1)$ are also eligible solutions.

Now $f(0, 1) = f(0, -1) = -1$ and $f(1, 0) = f(-1, 0) = 1$ so that minimum and maximum values are -1 and 1.

Finding global extrema of $f : \mathbb{R}^2 \to \mathbb{R}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $U \subset \mathbb{R}^2$ be a region with smooth closed boundary curve C. To find global emtremum of f in U :

- Locate all critical points of f in U.
- Find eligible global extremum of f on the curve C by using Lagrange multipliers or parametrization.
- Choose points among eligible solutions in C and the critical points at which f attains extreme values. These extreme values are global extremum.

Example: global extrema

Find global maximum and global minimum of the function $f(x,y) := (x^2+y^2)/2$ such that $x^2/2+y^2 \leq 1$.

Since $f_x = x$ and $f_y = y$, $(0,0)$ is the only critical point.

Next, consider $L(x,y,\lambda) := (x^2+y^2)/2 - \lambda(x^2/2+y^2-1)$. Then Lagrange multiplier equations are

$$x = \lambda x, \quad y = 2\lambda y, \quad x^2/2+y^2 = 1.$$

If $x=0$ then $y=\pm 1$ and $\lambda = 1/2$. If $y=0$ then $x = \pm\sqrt{2}$ and $\lambda = 1$. If $xy \neq 0$ then $\lambda = 1$ and $\lambda = 1/2$ -which is not possible.

Thus $(0,\pm 1)$ and $(\pm\sqrt{2},0)$ are eligible solutions for the boundary curve. We have $f(0,\pm 1) = 1/2$, $f(\pm\sqrt{2},0) = 1$ and $f(0,0) = 0$.

Test for constrained extrema of $f : \mathbb{R}^n \to \mathbb{R}$

Theorem: Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Suppose that f has an extremum at $\mathbf{p} \in U$ such that $g(\mathbf{p}) = \alpha$ and $\nabla g(\mathbf{p}) \neq \mathbf{0}$. Then there is a $\lambda \in \mathbb{R}$ such that $\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p})$.

Proof: Implicit function theorem converts constrained extremum to unconstrained extremum in a neighbourhood of $\mathbf{p}$. We omit the proof.

If $g(\mathbf{x}) = \alpha$ is a closed surface then global extremum is obtained by finding all points where $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ and choosing those where f is largest or smallest.

The Lagrangian is given by $L(\mathbf{x}, \lambda) := f(\mathbf{x}) - \lambda(g(\mathbf{x}) - \alpha)$. So, the multiplier equations are $\nabla L(\mathbf{x}, \lambda) = \mathbf{0}$.

Example

Maximize the function $f(x, y, z) := x + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

The multiplier equations are

$$1 = 2x\lambda, \quad 0 = 2y\lambda, \quad 1 = 2z\lambda, \quad x^2 + y^2 + z^2 = 1.$$

From first and 3rd equation, $\lambda \neq 0$. Thus, by second equation, $y = 0$. By first and 3rd equations $x = z \Rightarrow x = z = \pm 1/\sqrt{2}$.

Hence $\mathbf{p} := (1/\sqrt{2}, 0, 1/\sqrt{2})$ and $\mathbf{q} := (-1/\sqrt{2}, 0, -1/\sqrt{2})$ are eligible solutions. This shows that $f(\mathbf{p}) = \sqrt{2}$ and $f(\mathbf{q}) = -\sqrt{2}$.

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