Lecture 8:
Maxima and Minima

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Local extremum of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be continuous, where U is open. Then

- f has a local maximum at $\mathbf{p}$ if there exists $r > 0$ such that

$$f(\mathbf{x}) \leq f(\mathbf{p}) \text{ for } \mathbf{x} \in B(\mathbf{p}, r).$$

- f has a local minimum at $\mathbf{a}$ if there exists $\epsilon > 0$ such that

$$f(\mathbf{x}) \geq f(\mathbf{p}) \text{ for } \mathbf{x} \in B(\mathbf{p}, \epsilon).$$

A local maximum or a local minimum is called a local extremum.

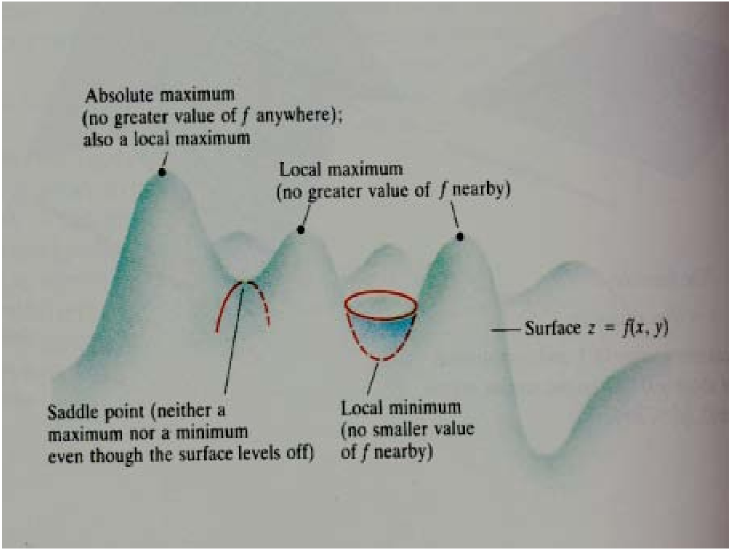

Figure: Local extremum of $z = f(x, y)$

Necessary condition for extremum of $\mathbb{R}^n \to \mathbb{R}$

Critical point: A point $\mathbf{p} \in U$ is a critical point of f if

$$\nabla f(\mathbf{p}) = 0.$$

Thus, when f is differentiable, the tangent plane to $z = f(\mathbf{x})$ at $(\mathbf{p}, f(\mathbf{p}))$ is horizontal.

Theorem: Suppose that f has a local extremum at $\mathbf{p}$ and that $\nabla f(\mathbf{p})$ exists. Then $\mathbf{p}$ is a critical point of f, i.e, $\nabla f(\mathbf{p}) = \mathbf{0}$.

Example: Consider $f(x, y) = x^2 - y^2$. Then $f_x = 2x = 0$ and $f_y = -2y = 0$ show that $(0, 0)$ is the only critical point of f. But $(0, 0)$ is not a local extremum of f.

Saddle point

Saddle point: A critical point of f that is not a local extremum is called a saddle point of f.

Examples:

- The point $(0,0)$ is a saddle point of $f(x,y) = x^2 - y^2$.
- Consider $f(x,y) = x^2y + y^2x$. Then $f_x = 2xy + y^2 = 0$ and $f_y = 2xy + x^2 = 0$ show that $(0,0)$ is the only critical point of f.

  But $(0,0)$ is a saddle point. Indeed, on $y = x$, f is both positive and negative near $(0,0)$.

Sufficient condition for extremum of $f : \mathbb{R}^2 \to \mathbb{R}$

Theorem: Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be C^2 and $\mathbf{p} \in U$ be a critical point, i.e, $f_x(\mathbf{p}) = 0 = f_y(\mathbf{p})$. Let

$$D := \det\left(\begin{bmatrix} f_{xx}(\mathbf{p}) & f_{xy}(\mathbf{p}) \\ f_{yx}(\mathbf{p}) & f_{yy}(\mathbf{p}) \end{bmatrix}\right) = f_{xx}(\mathbf{p})f_{yy}(\mathbf{p}) - f_{xy}^2(\mathbf{p}).$$

- If $f_{xx}(\mathbf{p}) > 0$ and $D > 0$ then f has a local minimum at $\mathbf{p}$.
- If $f_{xx}(\mathbf{p}) < 0$ and $D > 0$ then f has a local maximum at $\mathbf{p}$.
- If $D < 0$ then $\mathbf{p}$ is a saddle point.
- If $D = 0$ then nothing can be said.

Example

Find the minimum distance from the point $(1, 2, 0)$ to the cone $z^2 = x^2 + y^2$.

We minimize the square of distance

$$\begin{aligned} d^2 &= (x-1)^2 + (y-2)^2 + z^2 \\ &= (x-1)^2 + (y-2)^2 + x^2 + y^2 \\ &= 2x^2 + 2y^2 - 2x - 4y + 5. \end{aligned}$$

Consider $f(x, y) := 2x^2 + 2y^2 - 2x - 4y + 5$. Then $f_x = 4x - 2$ and $f_y = 4y - 4 \Rightarrow p := (1/2, 1)$ is the critical point.

Now $D = f_{xx}(p)f_{yy}(p) - f_{xy}^2(p) = 16 > 0$ and $f_{xx}(p) = 4 > 0$ $\Rightarrow f(p)$ is the minimum $\Rightarrow d = \sqrt{f(p)} = \sqrt{5/2}$.

Proof of sufficient condition for extremum

Write $H_f(\mathbf{p}) > 0$ to denote $f_{xx}(\mathbf{p}) > 0$ and $D(\mathbf{p}) > 0$, where

$$H_f(\mathbf{p}) := \begin{bmatrix} f_{xx}(\mathbf{p}) & f_{xy}(\mathbf{p}) \\ f_{yx}(\mathbf{p}) & f_{yy}(\mathbf{p}) \end{bmatrix} \text{ and } D(\mathbf{p}) := f_{xx}(\mathbf{p})f_{yy}(\mathbf{p}) - f_{xy}^2(\mathbf{p}).$$

Then

1. $H_f(\mathbf{p}) > 0 \Rightarrow H_f(\mathbf{p}+\mathbf{h}) > 0$ for $\|\mathbf{h}\| < \epsilon$.
2. $H_f(\mathbf{p}) > 0 \Rightarrow \langle H_f(\mathbf{p})\mathbf{h}, \mathbf{h}\rangle > 0$ for all $\mathbf{h} \neq 0$. Indeed,

$$\begin{aligned} \langle H_f(\mathbf{p})\mathbf{h}, \mathbf{h}\rangle &= h^2 f_{xx}(\mathbf{p}) + 2f_{xy}(\mathbf{p})hk + f_{yy}(\mathbf{p})k^2 \\ &= [(f_{xx}(\mathbf{p})h + f_{xy}(\mathbf{p})k)^2 + k^2 D(\mathbf{p})]/f_{xx}(\mathbf{p}). \end{aligned}$$

3. By EMVT there exists $0 < \theta < 1$ such that $f(\mathbf{p}+\mathbf{h}) - f(\mathbf{p}) = \frac{1}{2}\langle H_f(\mathbf{p}+\theta\mathbf{h})\mathbf{h}, \mathbf{h}\rangle > 0.$ ■

Sufficient condition for extremum of $f : \mathbb{R}^n \to \mathbb{R}$

Theorem: Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^2 and $\mathbf{p} \in U$ be a critical point, i.e, $\nabla f(\mathbf{p}) = \mathbf{0}$. Consider the Hessian

$$H_f(\mathbf{p}) := \begin{bmatrix} \partial_1\partial_1 f(\mathbf{p}) & \cdots & \partial_n\partial_1 f(\mathbf{p}) \\ \vdots & \cdots & \vdots \\ \partial_1\partial_n f(\mathbf{p}) & \cdots & \partial_n\partial_n f(\mathbf{p}) \end{bmatrix}.$$

- If $H_f(\mathbf{p}) > 0$ (positive definite) then f has a local minimum at $\mathbf{p}$.
- If $H_f(\mathbf{p}) < 0$ (negative definite) then f has a local maximum at $\mathbf{p}$.
- If $H_f(\mathbf{p})$ indefinite then $\mathbf{p}$ is a saddle point.
- If $\det H_f(\mathbf{p}) = 0$ then nothing can be said.

Positive definite matrix

Let A be a symmetric matrix of size n. Then A is said to be

- positive definite ($A > 0$) if $x^\top Ax > 0$ for nonzero $x \in \mathbb{R}^n$,
- negative definite ($A < 0$) if $x^\top Ax < 0$ for nonzero $x \in \mathbb{R}^n$,
- indefinite if $\det(A) \neq 0$ and there exits $x, y \in \mathbb{R}^n$ such that $x^\top Ax > 0$ and $y^\top Ay < 0$.

Fact: If $A > 0$ then there exists $\alpha > 0$ such that

$$x^\top Ax \geq \alpha \|x\|^2 \text{ for all } x \in \mathbb{R}^n.$$

Proof: $S := \{x \in \mathbb{R}^n : \|x\| = 1\}$ compact and $f(x) := x^\top Ax$ continuous $\Rightarrow \alpha := f(x_{\min}) = \min_{u \in S} f(u) > 0$. ■

Characterization of positive definite matrices

Fact: Let A_j denote the leading j-by-j principal sub-matrix of A for $j = 1, 2, \ldots, n$. Thus $\det(A_j)$ is the j-th principal minor.

Then

- $A > 0 \iff \det(A_j) > 0$ for $j = 1, 2, \ldots, n$, $\iff$ eigenvalues of A are positive.
- $A < 0 \iff \det(A_j) < 0$ for $j = 1, 2, \ldots, n$, $\iff$ eigenvalues of A are negative.
- A is indefinite $\iff \det(A) \neq 0$ and A has positive and negative eigenvalues.

Proof of sufficient condition for extremum of $f : \mathbb{R}^n \to \mathbb{R}$

Since f is C^2, we have

$$f(\mathbf{p}+\mathbf{h}) = f(\mathbf{p}) + \nabla f(\mathbf{p}) \bullet \mathbf{h} + \frac{1}{2}\mathbf{h} \bullet (H_f(\mathbf{p})\mathbf{h}) + e(\mathbf{h})\|\mathbf{h}\|^2,$$

where $e(\mathbf{h}) \to 0$ as $\mathbf{h} \to 0$.

1. $H_f(\mathbf{p}) > 0 \Rightarrow \mathbf{h} \bullet (H_f(\mathbf{p})\mathbf{h}) \geq \alpha\|\mathbf{h}\|^2$ for some $\alpha > 0$.
2. There exists $\delta > 0$ such that $\|\mathbf{h}\| < \delta \Rightarrow |e(\mathbf{h})| < \alpha/4$.
3. $f(\mathbf{p}+\mathbf{h}) - f(\mathbf{p}) = \frac{1}{2}\mathbf{h} \bullet (H_f(\mathbf{p})\mathbf{h}) + e(\mathbf{h})\|\mathbf{h}\|^2 \geq \frac{\alpha}{4}\|\mathbf{h}\|^2$ when $\|\mathbf{h}\| < \delta$.

Proof for saddle point of $f : \mathbb{R}^n \to \mathbb{R}$

If $H_f(\mathbf{p})$ is indefinite then there exists nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathbf{u} \bullet (H_f(\mathbf{p})\mathbf{u}) > 0 \text{ and } \mathbf{v} \bullet (H_f(\mathbf{p})\mathbf{v}) < 0.$$

Then $\phi(t) := f(\mathbf{p} + t\mathbf{u})$ has minimum at $t = 0$ whereas $\psi(t) := f(\mathbf{p} + t\mathbf{v})$ has a maximum at $t = 0$. Indeed,

$$\phi''(0) = \frac{\mathrm{d}^2 f(\mathbf{p} + t\mathbf{u})}{\mathrm{d}t^2}\Big|_{t=0} = \mathbf{u} \bullet (H_f(\mathbf{p})\mathbf{u}) > 0$$

and

$$\psi''(0) = \frac{\mathrm{d}^2 f(\mathbf{p} + t\mathbf{v})}{\mathrm{d}t^2}\Big|_{t=0} = \mathbf{v} \bullet (H_f(\mathbf{p})\mathbf{v}) < 0.$$

Example

Find the maxima, minima and saddle points of $f(x,y) := (x^2 - y^2)e^{-(x^2+y^2)/2}$. We have

$$\begin{aligned} f_x &= [2x - x(x^2 - y^2)]e^{-(x^2+y^2)/2} = 0, \\ f_y &= [-2y - y(x^2 - y^2)]e^{-(x^2+y^2)/2} = 0, \end{aligned}$$

so the critical points are $(0,0), (\pm\sqrt{2}, 0)$ and $(0, \pm\sqrt{2})$.

Point	f_{xx}	f_{xy}	f_{yy}	D	Type —
$(0,0)$	2	0	-2	-4	saddle
$(\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$	maximum
$(-\sqrt{2},0)$	$-4/e$	0	$-4/e$	$16/e^2$	maximum
$(0,\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$	minimum
$(0,-\sqrt{2})$	$4/e$	0	$4/e$	$16/e^2$	minimum

Example: global extrema

Find global maximum and global minimum of the function $f : [-2, 2] \times [-2, 2] \to \mathbb{R}$ given by $f(x, y) := 4xy - 2x^2 - y^4$.

To find global extrema, find extrema of f in the interior and then on the boundary.

Solving $f_x = 4y - 4x = 0$ and $f_y = 4x - 4y^3 = 0$ we obtain the critical points $(0, 0), (1, 1)$ and $(-1, -1)$. We have $f(1, 1) = f(-1, -1) = 1$. $(0, 0)$ is a saddle point.

For the boundary, consider $f(x, 2), f(x, -2), f(2, y), f(-2, y)$ and find their extrema on $[-2, 2]$. The global minimum is attained at $(2, -2)$ and $(-2, 2)$ with $f(2, -2) = -40$. The global maximum is attained at $(1, -1)$ and $(-1, 1)$.

*** End ***