Lecture 4: Partial and Directional derivatives, Differentiability

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Differential Calculus

Task: Extend differential calculus to the functions:

Case I: $f : A \subset \mathbb{R}^n \to \mathbb{R}$

Case II: $f : A \subset \mathbb{R} \to \mathbb{R}^n$

Case III: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$

Question: What does it mean to say that f is differentiable?

Parametric curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$

A continuous function $\mathbf{r} : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ is called a parametric curve in $\mathbb{R}^n$. The curve $\Gamma := \mathbf{r}([a, b])$ is parameterized by $\mathbf{r}(t)$.

Examples:

- $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ given by $\mathbf{r}(t) := \mathbf{a} + t\mathbf{b}$ parameterizes a line in $\mathbb{R}^n$ passing through $\mathbf{a}$ in the direction of $\mathbf{b}$.
- $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ given by $\mathbf{r}(t) := (\cos t, \sin t, t)$ parameterizes a circular helix.
- $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^2$ given by $\mathbf{r}(t) := (\cos t, \sin t)$ parameterizes the circle $x^2 + y^2 = 1$.

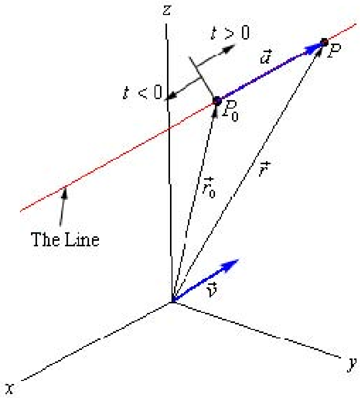

Figure: Line $\mathbf{r}(t) = \mathbf{p}_0 + t\mathbf{v}$

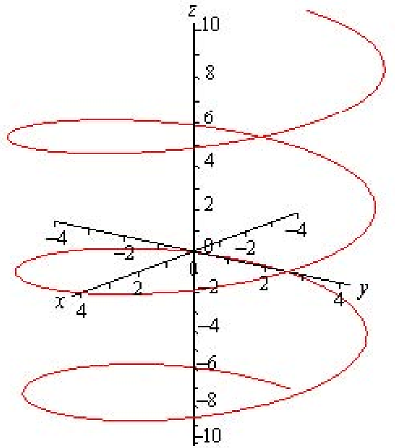

Figure: Helix $\mathbf{r}(t) = (4\cos t, 4\sin t, t)$

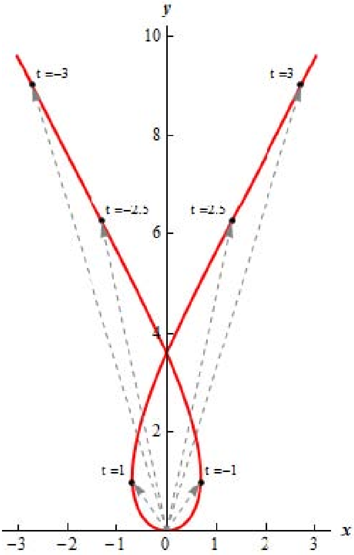

Figure: Plane curve $\mathbf{r}(t) = (t - 2\sin t, t^2)$

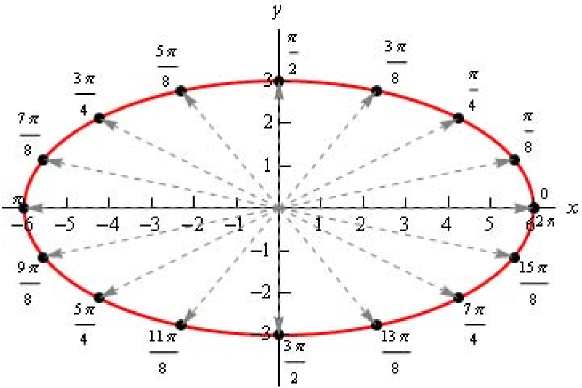

Figure: Ellipse $\mathbf{r}(t) = (6\cos t, 3\sin t)$

Differentiability of $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$

Definition: Let $\mathbf{r} : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ and $t_0 \in (a, b)$. If

$$\mathbf{r}'(t_0) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t_0) := \lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}$$

exists then $\mathbf{r}$ is differentiable at t_0. The derivative $\mathbf{r}'(t_0)$ is called the velocity vector.

Fact:

- $\mathbf{r}(t) = (r_1(t), \ldots, r_n(t))$, where $r_i : (a, b) \to \mathbb{R}$.
- $\mathbf{r}$ is differentiable at $t_0 \iff$ each r_i is differentiable at t_0, $i = 1, 2, \ldots, n$. Further, $\mathbf{r}'(t_0) = (r_1'(t_0), \ldots, r_n'(t_0))$.
- $\mathbf{r}$ differentiable at $t_0 \Rightarrow \mathbf{r}$ continuous at t_0.

Sum and product rules

Fact: Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable at $t_0 \in (a, b)$. Then for $\alpha \in \mathbb{R}$

1. $f + g$ and αf are differentiable at t_0. Further, $(f + g)'(t) = f'(t_0) + g'(t_0)$ and $(\alpha f)'(t_0) = \alpha f'(t_0)$.
2. $f \bullet g$ defined by $(f \bullet g)(t) := \langle f(t), g(t) \rangle$ is differentiable at t_0 and

$$(f \bullet g)'(t_0) = f'(t_0) \bullet g(t_0) + f(t_0) \bullet g'(t_0).$$

Velocity and tangent vectors

Let $\mathbf{r} : (a, b) \to \mathbb{R}^n$ be differentiable. Then treating $\mathbf{r}(t)$ as the position of a moving object at time t, we have

$$\text{scaled secant } = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \to \mathbf{r}'(t) \text{ as } \Delta t \to 0.$$

But scaled secant $\to$ tangent vector to the curve at $\mathbf{r}(t)$ as $\Delta t \to 0$.

Thus velocity vector $\mathbf{v}(t) := \mathbf{r}'(t)$ is tangent to the curve at $\mathbf{r}(t)$.

If $\mathbf{r}(t) := (\cos t, \sin t)$ then $\mathbf{v}(t) = \mathbf{r}'(t) = (-\sin t, \cos t)$.

Partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Then

$$\frac{\partial f}{\partial x}(a, b) := \lim_{t \to 0} \frac{f(a+t, b) - f(a, b)}{t},$$

when exists, is called partial derivative of f at (a, b) w.r.t to the first variable.

Other notations for $\dfrac{\partial f}{\partial x}(a, b)$:

$$f_x(a, b),\ \partial_x f(a, b),\ \partial_1 f(a, b).$$

Partial derivative $\dfrac{\partial f}{\partial y}(a, b)$ w.r.t. the second variable is defined similarly.

Partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. Then

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) := \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{e}_i) - f(\mathbf{a})}{t},$$

when exists, is called partial derivative of f at $\mathbf{a}$ w.r.t to the i-th variable.

Other notations for $\dfrac{\partial f}{\partial x_i}(\mathbf{a})$:

$$f_{x_i}(\mathbf{a}), \ \partial_{x_i} f(\mathbf{a}), \ \partial_i f(\mathbf{a}).$$

If $\partial_i f(\mathbf{a})$ exists for $i = 1, 2, \ldots, n$, then f is said to have first order partial derivatives at $\mathbf{a}$.

Examples

• Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then

$$\partial_1 f(0,0) = \partial_2 f(0,0) = 0$$

even though f is NOT continuous at $(0,0)$.

• Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } x \neq 0, y \neq 0, \\ x\sin(1/x) & \text{if } x \neq 0, y = 0, \\ y\sin(1/y) & \text{if } x = 0, y \neq 0. \end{cases}$$

Then f is continuous at $(0,0)$ but neither $\partial_1 f(0,0)$ nor $\partial_2 f(0,0)$ exists.

Moral: Partial derivatives $\not\Rightarrow$ continuity $\not\Rightarrow$ Partial derivatives

Sum, product and chain rule

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. Suppose $\partial_i f(\mathbf{a})$ and $\partial_i g(\mathbf{a})$ exist. Then

- $\partial_i(\alpha f)(\mathbf{a}) = \alpha \partial_i f(\mathbf{a})$ for $\alpha \in \mathbb{R}$,
- $\partial_i(f + g)(\mathbf{a}) = \partial_i f(\mathbf{a}) + \partial_i g(\mathbf{a})$,
- $\partial_i(fg)(\mathbf{a}) = \partial_i f(\mathbf{a}) g(\mathbf{a}) + f(\mathbf{a}) \partial_i g(\mathbf{a})$.
- If $h : \mathbb{R} \to \mathbb{R}$ is differentiable at $f(\mathbf{a})$ then $\partial_i(h \circ f)(\mathbf{a})$ exists and $\partial_i(h \circ f)(\mathbf{a}) = h'(f(\mathbf{a})) \partial_i f(\mathbf{a})$.

Gradient of $f : \mathbb{R}^n \to \mathbb{R}$

Define $\phi_i : \mathbb{R} \to \mathbb{R}$ by $\phi_i(t) := f(\mathbf{a} + t\mathbf{e}_i)$. Then

$$\begin{aligned}\partial_i f(\mathbf{a}) &= \lim_{t\to 0}\frac{\phi_i(t)-\phi_i(0)}{t} = \phi_i'(0) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{a}+t\mathbf{e}_i)_{|t=0},\\ &= \text{rate of change of } f \text{ at } \mathbf{a} \text{ in the direction } \mathbf{e}_i.\end{aligned}$$

Suppose partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ exist at $\mathbf{a} \in \mathbb{R}^n$. Then the vector

$$\nabla f(\mathbf{a}) := (\partial_1 f(a), \ldots, \partial_n f(\mathbf{a})) \in \mathbb{R}^n$$

is called the gradient of f at $\mathbf{a}$.

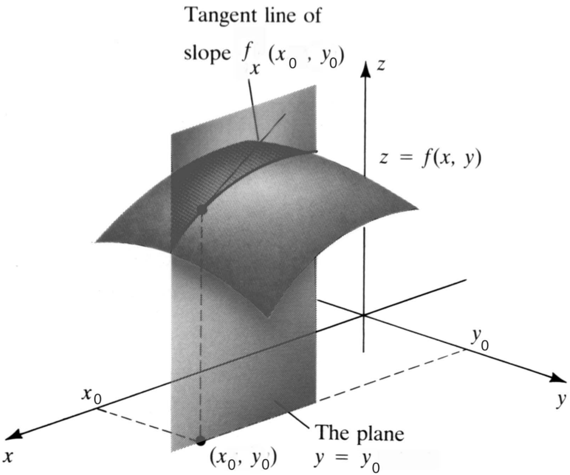

Figure: Graph of $z = f(x, y)$ and geometric interpretation of $\partial_x f(x_0, y_0)$.

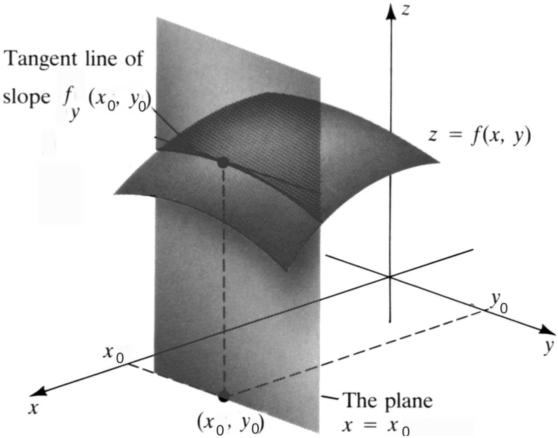

Figure: Graph of $z = f(x, y)$ and geometric interpretation of $\partial_y f(x_0, y_0)$.

Directional derivatives of $f : \mathbb{R}^n \to \mathbb{R}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. Also let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. Then the limit, when exists,

$$\begin{aligned} \mathrm{D}_{\mathbf{u}} f(\mathbf{a}) &:= \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{a} + t\mathbf{u})_{|t=0}, \\ &= \text{rate of change of } f \text{ at } \mathbf{a} \text{ in the direction } \mathbf{u}, \end{aligned}$$

is called directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{u}$.

- $\mathrm{D}_{\mathbf{u}} f(\mathbf{a})$, also denoted by $\dfrac{\partial f}{\partial \mathbf{u}}(\mathbf{a})$, is the rate of change of f at $\mathbf{a}$ in the direction $\mathbf{u}$.

Properties of directional derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. Also let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$.

Then

- Sum, product and chain rule similar to those of $\partial_i f(\mathbf{a})$ hold for $\mathrm{D}_{\mathbf{u}} f(\mathbf{a})$.
- If $\mathrm{D}_{\mathbf{u}} f(\mathbf{a})$ exists for all nonzero $\mathbf{u} \in \mathbb{R}^n$ then f is said to have directional derivatives in all directions.
- Obviously $\partial_i f(\mathbf{a}) = \mathrm{D}_{\mathbf{e}_i} f(\mathbf{a})$. Hence $\mathrm{D}_{\mathbf{u}} f(\mathbf{a})$ exists in all directions $\mathbf{u} \Rightarrow \partial_i f(\mathbf{a})$ exist for $i = 1, 2, \ldots, n$.

Examples

1. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := \sqrt{|xy|}$. Then $\partial_1 f(0, 0) = 0 = \partial_2 f(0, 0)$ and f is continuous at $(0, 0)$. However, $\mathrm{D}_{\mathbf{u}} f(0, 0)$ does NOT exist for $u_1 u_2 \neq 0$.

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0, 0) = 0$ and $f(x, y) := \dfrac{x^2 y}{x^4 + y^2}$ if $(x, y) \neq (0, 0)$. Then f is NOT continuous at $(0, 0)$, $\partial_1 f(0, 0) = 0 = \partial_2 f(0, 0)$ and $\mathrm{D}_{\mathbf{u}} f(0, 0)$ exits for all $\mathbf{u}$. Further, $\mathrm{D}_{\mathbf{u}} f(0, 0) = u_1^2 / u_2$ for $u_1 u_2 \neq 0$.

Moral: Partial derivatives $\not\Rightarrow$ Directional derivative $\not\Rightarrow$ Continuity $\not\Rightarrow$ Directional derivative.

*** End ***