Lecture 3:
Limit and Continuity of Functions

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Topology of $\mathbb{R}^n$

Open Ball: Let $\epsilon > 0$ and $\mathbf{a} \in \mathbb{R}^n$. Then

$$B(\mathbf{a}, \epsilon) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \epsilon\}$$

is called open ball of radius ϵ centred at $\mathbf{a}$.

Open set: $O \subset \mathbb{R}^n$ is open if for any $\mathbf{x} \in O$ there is $\epsilon > 0$ such that $B(\mathbf{x}, \epsilon) \subset O$.

Examples:

1. $B(\mathbf{a}, \epsilon) \subset \mathbb{R}^n$ is an open set.
2. $O := (a_1, b_1) \times \cdots \times (a_n, b_n)$ is open in $\mathbb{R}^n$.
3. $\mathbb{R}^n$ is open.

Closed set: $S \subset \mathbb{R}^n$ is closed if $S^c := \mathbb{R}^n \setminus S$ is open.

Examples:

1. $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ is closed set.
2. $E := [a_1, b_1] \times \cdots \times [a_n, b_n]$ is closed in $\mathbb{R}^n$.
3. $\mathbb{R}^n$ is closed.

Theorem: Let $S \subset \mathbb{R}^n$. Then the following are equivalent:

1. S is closed.
2. If $(\mathbf{x}_k) \subset S$ and $\mathbf{x}_k \to \mathbf{x} \in \mathbb{R}^n$ then $\mathbf{x} \in S$.

Limit point: Let $A \subset \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$. Then $\mathbf{a}$ is a limit point of A if $A \cap (B(\mathbf{a}, \epsilon) \setminus \{\mathbf{a}\}) \neq \emptyset$ for any $\epsilon > 0$.

Examples:

1. Each point in $B(\mathbf{a}, \epsilon)$ is a limit point.
2. Each $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x} - \mathbf{a}\| = \epsilon$ is a limit point of $B(\mathbf{a}, \epsilon)$.

Fact: Let $S \subset \mathbb{R}^n$. Then S is closed $\iff$ S contains all of its limit points.

Limit of a function

Definition:

- Let $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^n$ and $L \in \mathbb{R}$. Then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Longrightarrow |f(\mathbf{x}) - L| < \epsilon.$$

- Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $L \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{R}^n$ be a limit point of A. Then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\mathbf{x} \in A \text{ and } 0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Longrightarrow |f(\mathbf{x}) - L| < \epsilon.$$

Sequential characterization

Theorem: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$, $L \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$ be a limit point of A. Then the following are equivalent:

- $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$
- If $(\mathbf{x}_k) \subset A \setminus \{\mathbf{a}\}$ and $\mathbf{x}_k \to \mathbf{a}$ then $f(\mathbf{x}_k) \to L$.

Proof:

Remark:

- Limit, when exists, is unique.
- Sum, product and quotient rules hold.

Examples:

1. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

Then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Iterated limit

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$. Then $\lim_{x\to a} \lim_{y\to b} f(x, y)$, when exists, is called an iterated limit of f at (a, b).

Ditto for $\lim_{y\to b} \lim_{x\to a} f(x, y)$ when it exists.

Remark:

- Iterated limits are defined similarly for $f : A \subset \mathbb{R}^n \to \mathbb{R}$.
- Existence of limit does not guarantee existence of iterated limits and vice-versa.
- Iterated limits when exist may be unequal. However, if limit and iterated limits exist then they are all equal.

Examples:

1. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

   Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) := 0$ and $f(x,y) := \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Then iterated limits exist at $(0,0)$ and are unequal.

Continuous function and open set

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{a} \in \mathbb{R}^n$. Then

- f continuous at $\mathbf{a}$ $\iff$ for any open set $f(\mathbf{a}) \in V \subset \mathbb{R}^m$ there is an open set $\mathbf{a} \in U \subset \mathbb{R}^n$ such that $U \subset f^{-1}(V)$.
- f is continuous at $\mathbf{a}$ $\iff$ $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.
- f is continuous on $\mathbb{R}^n$ $\iff$ $f^{-1}(V)$ is open in $\mathbb{R}^n$ whenever $V \subset \mathbb{R}^m$ is open.

Example: Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $c \in \mathbb{R}$. Then

- $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < c\}$ is open
- $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq c\}$ is closed

Continuous function and compact set

Compact set: $A \subset \mathbb{R}^n$ is compact if $(\mathbf{x}_k) \subset A$ then $(\mathbf{x}_k)$ has a subsequence $(\mathbf{x}_{k_p})$ that converges to some $\mathbf{x} \in A$.

Examples:

1. $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ is compact.
2. $E := [a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact.

Bounded set: $S \subset \mathbb{R}^n$ is bounded if $S \subset B(0, \alpha)$ for some $\alpha > 0$.

Theorem (Heine-Borel):
$S \subset \mathbb{R}^n$ is compact $\iff$ S is closed and bounded.

Extreme Value Theorem

Theorem: Let $f : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous. If K is compact then

- $f(K)$ is compact,
- f is uniformly continuous on K.

Theorem (EVT): Let $f : K \subset \mathbb{R}^n \to \mathbb{R}$ be continuous and K compact. Then

- there is $\mathbf{x}_{\min} \in K$ such that $f(\mathbf{x}_{\min}) = \inf\{f(\mathbf{x}) : \mathbf{x} \in K\}$,
- there is $\mathbf{x}_{\max} \in K$ such that $f(\mathbf{x}_{\max}) = \sup\{f(\mathbf{x}) : \mathbf{x} \in K\}$.

*** End ***