Lecture 2: Continuous functions

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Continuous functions

Task: Analyze continuity of the functions:

Case I: $f : A \subset \mathbb{R}^n \to \mathbb{R}$

Case II: $f : A \subset \mathbb{R} \to \mathbb{R}^n$

Case III: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$

Question: What does it mean to say that f is continuous?

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then

- $\mathbb{R} \to \mathbb{R}$, $x \mapsto f(x,y)$ is continuous for each fixed y
- $\mathbb{R} \to \mathbb{R}$, $y \mapsto f(x,y)$ is continuous for each fixed x

Is f continuous at $(0,0)$?

Continuity of $f : \mathbb{R}^n \to \mathbb{R}$

Definition 1: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. Then

- f continuous at $\mathbf{a}$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{a}\| < \delta \Longrightarrow |f(\mathbf{x}) - f(\mathbf{a})| < \epsilon.$$

- f is continuous on $\mathbb{R}^n$ if f is continuous at each $\mathbf{x} \in \mathbb{R}^n$.

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) := 0$ and $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Then f is NOT continuous at $(0,0)$.

Moral: Let $f : \mathbb{R}^2 \to \mathbb{R}$. Then continuity of $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ do not guarantee continuity of f.

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x\sin(1/y) + y\sin(1/x) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

Then f is continuous at $(0,0)$. But

- $x \mapsto f(x,y)$ is NOT continuous at 0 for $y \neq 0$
- $y \mapsto f(x,y)$ is NOT continuous at 0 for $x \neq 0$

Remark: Let $f : \mathbb{R}^2 \to \mathbb{R}$. Then continuity of f at (a,b) does NOT imply continuity of $t \mapsto f(t,y)$ and $s \mapsto f(x,s)$ at a and b, respectively, for each (x,y).

Let $\mathbb{S} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Define $f : \mathbb{S} \to \mathbb{R}$ by $f(x, y, z) := x - y + z$. Is f continuous?

Definition 2: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in A$. Then

- f continuous at $\mathbf{a}$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\mathbf{x} \in A \text{ and } \|\mathbf{x} - \mathbf{a}\| < \delta \implies |f(\mathbf{x}) - f(\mathbf{a})| < \epsilon.$$

- f is continuous on A if f is continuous at each $\mathbf{x} \in A$.

Example: Let $A := \mathbb{S} \cup \{(0, 0, 0)\}$. Consider $f : A \to \mathbb{R}$ given by $f(0, 0, 0) := 1$ and $f(x, y, z) := x + y + z$ for $(x, y, z) \in \mathbb{S}$. Then f is continuous on A.

Sequential characterization

Theorem: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in A$. Then the following are equivalent:

- f is continuous at $\mathbf{a}$
- If $(\mathbf{x}_k) \subset A$ and $\mathbf{x}_k \to \mathbf{a}$ then $f(\mathbf{x}_k) \to f(\mathbf{a})$.

Proof:

Examples: Examine continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

1. $f(x, y) := \sin(xy)$.
2. $f(x, y) := e^{x^2+y^2}$
3. $f(0, 0) := 0$ and $f(x, y) := \dfrac{x^2 y}{x^4 + y^2}$ for $(x, y) \neq (0, 0)$.

Sum, product and composition

Theorem: Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ be continuous at $\mathbf{a} \in A$. Then

- $f + g : A \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) + g(\mathbf{x})$ is continuous at $\mathbf{a}$,
- $f \cdot g : A \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x})g(\mathbf{x})$ is continuous at $\mathbf{a}$,
- If $h : \mathbb{R} \to \mathbb{R}$ is continuous at $g(\mathbf{a})$ then

  $h \circ g : A \to \mathbb{R}, \mathbf{x} \mapsto h(g(\mathbf{x}))$ is continuous at $\mathbf{a}$.

Proof: Use sequential characterization.

Examples: Examine continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

1. $f(x, y) := e^x \sin(x^2 y) + e^{xy+1} + \sqrt{x^2 + y^2}$,
2. $f(0, 0) := 0$ and $f(x, y) := \dfrac{xy}{\sqrt{x^2 + y^2}}$ for $(x, y) \neq (0, 0)$,
3. $f(0, 0) := 0$ and $f(x, y) := \dfrac{\sin^2(x - y)}{|x| + |y|}$ for $(x, y) \neq (0, 0)$.

Continuity of $f : \mathbb{R}^n \to \mathbb{R}^m$

Definition 3: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{a} \in A$. Then

- f continuous at $\mathbf{a}$ if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\mathbf{x} \in A \text{ and } \|\mathbf{x} - \mathbf{a}\| < \delta \Longrightarrow \|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon.$$

- f is continuous on A if f is continuous at each $\mathbf{x} \in A$.

Examples: Examine continuity of $f : \mathbb{R}^n \to \mathbb{R}^m$ given by

1. $f(x) := (\sin(x), \cos(x), x)$
2. $f(x, y) := (e^x \sin(y), y\cos(x), x^3 + y)$
3. $f(x, y, z) := (\dfrac{\sin(x - y)}{1 + |x| + |y|}, e^{x^2 - y^2 - z^2})$.

Componentwise continuity characterization

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$$

where $f_i : A \to \mathbb{R}$.

Theorem: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})).$$

Then f is continuous at $\mathbf{a} \in A \iff f_i$ is continuous at $\mathbf{a}$ for $i = 1, 2, \ldots, m$.

Proof: Use $|f_i(\mathbf{x})| \leq \|f(\mathbf{x})\|$ and $\|f(\mathbf{x})\| \leq \sum_{i=1}^m |f_i(\mathbf{x})|$.

Sum, product and composition

Theorem: Let $f, g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous at $\mathbf{a} \in A$. Then

- $f + g : A \to \mathbb{R}^m, \mathbf{x} \mapsto f(\mathbf{x}) + g(\mathbf{x})$ is continuous at $\mathbf{a}$,
- $f \bullet g : A \to \mathbb{R}, \mathbf{x} \mapsto \langle f(\mathbf{x}), g(\mathbf{x}) \rangle$ is continuous at $\mathbf{a}$,
- If $h : \mathbb{R}^m \to \mathbb{R}^p$ is continuous at $g(\mathbf{a})$ then

  $h \circ g : A \to \mathbb{R}^p, \mathbf{x} \mapsto h(g(\mathbf{x}))$ is continuous at $\mathbf{a}$.

Proof: Use sequential characterization.

Uniform continuity of $f : \mathbb{R}^n \to \mathbb{R}$

Definition: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$. Then f is uniformly continuous on A if for any $\epsilon > 0$ there is $\delta > 0$ such that

$$\mathbf{x}, \mathbf{y} \in A \text{ and } \|\mathbf{x} - \mathbf{y}\| < \delta \Longrightarrow |f(\mathbf{x}) - f(\mathbf{y})| < \epsilon.$$

Example: The function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) := \|\mathbf{x}\|$ is uniformly continuous. What about $g(\mathbf{x}) := \|\mathbf{x}\|^2$?

Fact:

$(\mathbf{x}_k) \subset A$ Cauchy $+$ f uniformly cont. $\Longrightarrow (f(\mathbf{x}_k))$ Cauchy.

Sequential characterization

Theorem: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$. Then the following are equivalent:

- f is uniformly continuous on A.
- If $(\mathbf{x}_k) \subset A$ and $(\mathbf{y}_k) \subset A$ such that $\|\mathbf{x}_k - \mathbf{y}_k\| \to 0$ then $|f(\mathbf{x}_k) - f(\mathbf{y}_k)| \to 0$.

Proof:

Examples:

1. $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + y^2$ is NOT uniformly continuous.
2. $f : (0, 1) \times (0, 1) \to \mathbb{R}, (x, y) \mapsto 1/(x + y)$ is NOT uniformly continuous.

Lipschitz continuity: $f : \mathbb{R}^n \to \mathbb{R}^m$

Definition: Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Then f is Lipschitz continuous on A if there is $M > 0$ such that

$$\mathbf{x}, \mathbf{y} \in A \Longrightarrow \|f(\mathbf{x}) - f(\mathbf{y})\| \leq M \|\mathbf{x} - \mathbf{y}\|.$$

Lipschitz continuity $\Longrightarrow$ Uniform Continuity $\Longrightarrow$ Continuity

Examples:

1. $f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \|\mathbf{x}\|$ is Lipschitz continuous.
2. $f : [0, 1] \to \mathbb{R}, x \mapsto \sqrt{x}$ is uniformly continuous but NOT Lipschitz.
3. $f : (0, 1) \to \mathbb{R}, x \mapsto 1/x$ is continuous but NOT uniformly continuous.

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