Lecture 13:
Multiple Integrals

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Riemann sum for double integral

Consider the rectangle $\mathbf{R} := [a, b] \times [c, d]$ and a bounded function $f : \mathbf{R} \to \mathbb{R}$.

Let P be a partition of $\mathbf{R}$ into mn sub-rectangles R_{ij} and $\mathbf{c}_{ij} \in R_{ij}$ for , $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Also let

$$\Delta A_{ij} = \mathrm{area}(R_{ij}) = \Delta x_i \Delta y_j \text{ and } \mu(P) := \max_{ij} \Delta A_{ij}.$$

Consider the Riemann sum

$$S(P, f) := \sum_{i=1}^{m} \sum_{j=1}^{n} f(\mathbf{c}_{ij}) \Delta A_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j.$$

Double integral

Definition: If $\lim_{\mu(P)\to 0} S(P,f)$ exists then f is said to be Riemann integrable and the (double) integral of f over $\mathbf{R}$ is given by

$$\iint_{\mathbf{R}} f(x,y)dA = \iint_{\mathbf{R}} f(x,y)dxdy = \lim_{\mu(P)\to 0} S(P,f).$$

- If $f(x,y) \geq 0$ then $\iint_{\mathbf{R}} f(x,y)dA$ gives the volume of the region bounded by $\mathbf{R}$ and the graph of f.

Theorem: If $f : \mathbf{R} \to \mathbb{R}$ is continuous then f is Riemann integrable over $\mathbf{R}$.

Theorem: Let $f, g : \mathbf{R} \to \mathbb{R}$ be Riemann integrable. Then

- $f + \alpha g$ is Riemann integrable for $\alpha \in \mathbb{R}$ and
$$\iint_{\mathbf{R}} (f + \alpha g) dA = \iint_{\mathbf{R}} f dA + \alpha \iint_{\mathbf{R}} g dA$$
- $|f|$ is Riemann integrable and
$$|\iint_{\mathbf{R}} f(x, y) dA| \leq \iint_{\mathbf{R}} |f(x, y)| dA.$$
- $\iint_{\mathbf{R}} dA = \mathrm{Area}(\mathbf{R})$.
- If $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$ then
$$\iint_{\mathbf{R}} f(x, y) dA = \iint_{\mathbf{R}_1} f(x, y) dA + \iint_{\mathbf{R}_2} f(x, y) dA.$$

Iterated integrals

Let $f : \mathbf{R} \to \mathbb{R}$. Suppose that for each fixed $x \in [a, b]$

$$\phi(x) := \int_c^d f(x, y) dy$$

exists. If ϕ is Riemann integrable on $[a, b]$ then

$$\int_a^b \phi(x) dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

is called an iterated integral of f over $\mathbf{R}$.

Similarly $\int_c^d \left(\int_a^b f(x, y) dx \right) dy$, when exists, is another iterated integral of f over $\mathbf{R}$.

Remark: Iterated integral, when exists, allows *integrate w.r.t. one variable at a time approach.* Unfortunately,

- an iterated integral may or may not exists even if f is Riemann integrable,
- iterated integrals, when exist, may be unequal.

Example: Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} 1 & x \text{ rational} \\ 2y & x \text{ irrational} \end{cases}$$

Then f is NOT Riemann integrable but

$$\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = 1.$$

Fubini's Theorem

Theorem: Let $f : \mathbf{R} \to \mathbb{R}$ be Riemann integrable. Suppose that for each fixed $x \in [a, b]$

$$\phi(x) := \int_c^d f(x, y)dy$$

exists. Then ϕ is Riemann integrable on $[a, b]$ and

$$\iint_{\mathbf{R}} f(x, y)dA = \int_a^b \phi(x)dx = \int_a^b \left(\int_c^d f(x, y)dy \right) dx.$$

Fubini's Theorem (cont.)

Similarly, suppose that for each fixed $y \in [c, d]$

$$\psi(y) := \int_a^b f(x, y)dx$$

exists. Then ψ is Riemann integrable on $[c, d]$ and

$$\iint_{\mathbf{R}} f(x, y)dA = \int_c^d \psi(y)dy = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

Fubini's Theorem for continuous functions

Theorem: Let $f : \mathbf{R} \to \mathbb{R}$ be continuous. Then both the iterated limits exist and

$$\begin{aligned}\iint_{\mathbf{R}} f(x,y)dA &= \int_a^b \left(\int_c^d f(x,y)dy\right) dx \\ &= \int_c^d \left(\int_a^b f(x,y)dx\right) dy.\end{aligned}$$

Example: Evaluate $\iint_{\mathbf{R}} xe^{xy}dA$, where $\mathbf{R} = [0,1] \times [0,1]$. Since the function is continuous,

$$\iint_{\mathbf{R}} xe^{xy}dA = \int_0^1 (\int_0^1 xe^{xy}dy)dx = \int_0^1 (e^x - 1)dx = e - 2.$$

Double integrals over general domains

Definition: Let $D \subset \mathbb{R}^2$ be bounded and $f : D \to \mathbb{R}$ be a bounded function. Then f is said to be integrable over D if for some rectangle $\mathbf{R}$ containing D the function

$$F(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable over $\mathbf{R}$. The double integral of f over D is then defined by

$$\iint_D f(x,y)dA := \iint_{\mathbf{R}} F(x,y)dA.$$

Remark: Since f is zero outside D the choice of $\mathbf{R}$ is unimportant in defining double integral of f over D.

Special domains

Definition: Let $D \subset \mathbb{R}^2$. Then D is called a Type-I domain if

$$D = \{(x, y) : x \in [a, b] \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$$

for some $[a, b] \subset \mathbb{R}$ and continuous functions $\phi_i : [a, b] \to \mathbb{R}$.

• Similarly, D is called a Type-II domain if

$$D = \{(x, y) : \psi_1(y) \leq x \leq \psi(y) \text{ and } y \in [c, d]\}$$

for some $[c, d] \subset \mathbb{R}$ and continuous functions $\psi_i : [a, b] \to \mathbb{R}$.

• Finally, D is called Type-III domain if D is simultaneously of Type-I and Type-II.

Double integral over special domains

Theorem: Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous. If D is Type-I and $D = \{(x, y) : x \in [a, b] \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$ then f is integrable over D and

$$\iint_D f(x, y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right) dx.$$

If D is Type-II and
$D = \{(x, y) : \psi_1(y) \leq x \leq \psi(y) \text{ and } y \in [c, d]\}$ then f is integrable over D and

$$\iint_D f(x, y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \right) dy.$$

Area and Volume

Let $D \subset \mathbb{R}^2$ be a special (Type-I or Type-II or Type-II) domain and $f : D \to \mathbb{R}$ be continuous. Then

$$\mathrm{Area}(D) = \iint_D dA.$$

If $f(x, y) \geq 0$ then the volume of the solid S bounded by D and the graph of $z = f(x, y)$ is given by

$$\mathrm{Volume}(S) = \iint_D f(x, y) dA.$$

Example

Find the volume of the solid S bounded by elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2, y = 2$, and the coordinate planes.

$$\begin{aligned} \text{Volume}(S) &= \iint_{\mathbf{R}} (16 - x^2 - 2y^2)\, dA \\ &= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy = 48. \end{aligned}$$

Example

Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

The region D is Type-I and

$$\begin{aligned}\iint_D (x+2y)dA &= \int_{-1}^{1}\left(\int_{2x^2}^{1+x^2}(x+2y)dy\right)dx \\ &= \int_{-1}^{1}(-3x^4 - x^3 + 2x^2 + x + 1)dx = \frac{32}{15}.\end{aligned}$$

Green's Theorem

Let $D \subset \mathbb{R}^2$ be a simply connected (no holes) region with positively oriented boundary ∂D. Let $F = (P, Q)$ be C^1 vector field on D. Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} (P(x,y)dx + Q(x,y)dy) = \oint_{\partial D} F \bullet \mathbf{dr}.$$

Divergence Theorem in $\mathbb{R}^2$: Considering $F := (P, Q)$, we have

$$\iint_D \mathrm{div}(F) dA = \oint_{\partial D} F \bullet \mathbf{n} ds,$$

where $\mathbf{n}$ is unit outward normal.

Applications of Green's Theorem

- Evaluation of area

$$\mathrm{Area}(D) = \iint_D dA = \frac{1}{2}\oint_{\partial D}(xdy - ydx).$$

- Let $f : D \to \mathbb{R}$ be C^2. Then for $F = (-f_y, f_x)$,

$$\begin{aligned}\iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dA &= \oint (-f_y dx + f_x dy) \\ &= \oint \nabla f \bullet \mathbf{n} ds = \oint \frac{\partial f}{\partial \mathbf{n}} ds,\end{aligned}$$

where $\mathbf{n}$ is unit outward normal.

Example

Let C be a circle of radius a centered at the origin. Find $\oint_C F \bullet \mathbf{dr}$ for $F = (-y, x)$ using Greens theorem.

$$\oint_C F \bullet \mathbf{dr} = \iint_D 2dA = 2\pi a^2.$$

*** End ***