Lecture 12: Line Integrals of vector fields

Rafikul Alam
Department of Mathematics
IIT Guwahati

Other types of line integrals of scalar fields

Let Γ be parametrized by PC^1 curve $\mathbf{r}(t) := (x_1(t), \ldots, x_n(t))$ and $f : \Gamma \to \mathbb{R}$ be continuous. Then the line integral of f along Γ w.r.t. x_i is defined by

$$\int_\Gamma f dx_i := \int_a^b f(x_1(t), \ldots, x_n(t)) x_i'(t) dt.$$

For $n = 3$, these integrals are denoted by

$$\int_\Gamma f(x,y,z)dx, \quad \int_\Gamma f(x,y,z)dy \text{ and } \int_\Gamma f(x,y,z)dz.$$

Line integrals of vector fields

Definition: Let Γ be a curve in $\mathbb{R}^n$ parametrized by a PC^1 path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ and let F be a continuous vector field on an open set containing Γ. Then the line integral of F over Γ is defined by

$$\begin{aligned}\int_\Gamma F \bullet \mathbf{dr} \; &:= \; \int_a^b F(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt = \int_a^b \langle F(\mathbf{r}(t)), \mathbf{r}'(t)\rangle dt \\ &= \; \lim_{\mu(P)\to 0} \sum_{j=1}^m F(\mathbf{p}_j) \bullet \Delta\mathbf{r}_j,\end{aligned}$$

where $\mu(P) = \max_j \|\Delta\mathbf{r}_j\|$.

Note that $[a, b] \longrightarrow \mathbb{R}, t \longmapsto F(\mathbf{r}(t)) \bullet \mathbf{r}'(t)$ is piecewise continuous and hence Riemann integrable.

Line integrals of vector fields via scalar fields

Suppose that $\mathbf{r}$ is (piecewise) smooth. Then $\|\mathbf{r}'(t)\| \neq 0$.
Define the tangent vector field $T(\mathbf{r}(t)) := \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ to Γ at $\mathbf{r}(t)$.

Then $F \bullet T$ is the tangential component of F and

$$\begin{aligned}
\int_\Gamma F \bullet \mathbf{dr} &= \int_a^b F(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt \\
&= \int_a^b F(\mathbf{r}(t)) \bullet T(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt \\
&= \int_\Gamma F \bullet T ds = \int_\Gamma \langle F,\ T \rangle ds.
\end{aligned}$$

Line integral of F = line integral of the scalar field $F \bullet T$.

Notations for line integrals of vector fields

- When Γ is closed, that is, $\mathbf{r}(a) = \mathbf{r}(b)$, the line integral

$$\int_\Gamma F \bullet \ \mathbf{dr} \text{ is denoted by } \oint_\Gamma F \bullet \mathbf{dr}.$$

- When $n = 2$ and $F = (P, Q)$ the line integral is written as

$$\int_\Gamma F \bullet \mathbf{dr} = \int_\Gamma (P(x,y)dx + Q(x,y)dy).$$

- For $n = 3$ and $F = (P, Q, R)$ the line integral is written as

$$\int_\Gamma F \bullet \mathbf{dr} = \int_\Gamma (P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz).$$

Examples

- Evaluate $\int_\Gamma F \bullet \mathbf{dr}$, where $F(x,y,z) := (xy, yz, zx)$ and $\mathbf{r}(t) := (t, t^2, t^2), t \in [0,1]$. We have

$$\int_\Gamma F \bullet \mathbf{dr} = \int_0^1 F(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt = \int_0^1 (t^3 + 5t^6)dt = \frac{27}{28}.$$

- Evaluate $\int_\Gamma (yx^2 dx + \sin(\pi y)dy)$, where Γ is the line segment from $(0,2)$ to $(1,4)$.

  We have $\mathbf{r}(t) = (t,\ 2+2t), t \in [0,1]$. Thus

$$\int_\Gamma (yx^2 dx + \sin(\pi y)dy) =$$

$$= \int_0^1 2\sin(\pi(2+2t))dt + \int_0^1 (2+2t)t^2 dt = \frac{7}{6}$$

Oriented path

- A parametrization $\mathbf{r} : [a, b] \to \mathbb{R}^n$ determines an orientation or a direction of the curve $\Gamma = \mathbf{r}([a, b])$. Indeed, as t varies from a to b, $\mathbf{r}(t)$ traverses the path from $\mathbf{r}(a)$ to $\mathbf{r}(b)$.

- $\int_\Gamma F \bullet \mathbf{dr}$ is invariant under equivalent parametrization of Γ.

- Let Γ be an oriented path. Denote the reverse orientation of Γ by $-\Gamma$. If $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is a parametrization of the oriented path Γ then $\rho : [a, b] \to \mathbb{R}^n$ given by $\rho(t) := \mathbf{r}(a + b - t)$ is a parametrization of $-\Gamma$.

- Let Γ be an oriented path. Then

$$\int_{-\Gamma} F \bullet \mathbf{dr} = -\int_\Gamma F \bullet \mathbf{dr}.$$

Work done

Definition: The work done by a force field F on a particle traversing an oriented path Γ is the line integral

$$\text{Work } = \int_\Gamma F \bullet \mathbf{dr}.$$

Remark: The total work done by F on a particle traversing the path Γ and then reversing back to the initial point is

$$W = \int_\Gamma F \bullet \mathbf{dr} + \int_{-\Gamma} F \bullet \mathbf{dr} = \int_\Gamma F \bullet \mathbf{dr} - \int_\Gamma F \bullet \mathbf{dr} = 0.$$

Example

Consider the gravitational force field $F = -\dfrac{mMG\mathbf{r}}{\|\mathbf{r}\|^3}$, where $\mathbf{r} := (x, y, z)$. Find the work done by F in moving a particle of mass m from point $(3, 4, 12)$ to the point $(1, 0, 0)$ along a piecewise smooth curve Γ.

Setting $f(x, y, z) := \frac{mMG}{\|\mathbf{r}\|}$, we have $F = \nabla f$. Consequently

$$W = \int_\Gamma \nabla f \bullet \mathbf{dr} = f(1, 0, 0) - f(3, 4, 12) = \frac{12mMG}{13}.$$

Fundamental Theorem for line integrals

If $f : [a, b] \to \mathbb{R}$ is C^1 then by FTI $\int_a^b f'(x)dx = f(b) - f(a)$.

Theorem: Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be C^1. Let $\mathbf{r} : [a, b] \to \mathbb{R}^n$ be PC^1 such that $\mathbf{r}([a, b]) \subset U$. Then

$$\int_\Gamma \nabla f \bullet \mathbf{dr} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof: We have

$$\begin{aligned} \int_\Gamma \nabla f \bullet \mathbf{dr} &= \int_a^b \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \end{aligned}$$

Consequence of FTLI

If F is a conservative vector field, that is, $F = \nabla f$ for some scalar field f, then $\int_\Gamma F \bullet \mathbf{dr}$ only depends on the end points of Γ and hence independent of the path Γ.

So, in particular, if Γ is closed then $\oint_\Gamma F \bullet \mathbf{dr} = 0$.

Generally $\int_\Gamma F \bullet \mathbf{dr}$ depends on the oriented path Γ and

$$\int_{\Gamma_1} F \bullet \mathbf{dr} \neq \int_{\Gamma_2} F \bullet \mathbf{dr}$$

for two curves having the same initial and final points.

Example

Consider the vector field $F(x, y, z) := (y, -x, 1)$ and the paths joining $(1, 0, 0)$ to $(1, 0, 1)$ given by

$$\begin{aligned} \Gamma_1 : \mathbf{r}(t) &= (\cos t, \sin t, \frac{t}{2\pi}),\ t \in [0, 2\pi], \\ \Gamma_2 : \mathbf{r}(t) &= (\cos t^3, \sin t^3, \frac{t^3}{2\pi}),\ t \in [0, \sqrt[3]{2\pi}], \\ \Gamma_3 : \mathbf{r}(t) &= (\cos t, -\sin t, \frac{t}{2\pi}),\ t \in [0, 2\pi]. \end{aligned}$$

Example (contd.)

Then

$$\begin{aligned}
\int_{\Gamma_1} F \bullet \mathbf{dr} &= \int_0^{2\pi} (-\sin^2 t - \cos^2 t + 1/2\pi) dt = 1 - 2\pi \\
\int_{\Gamma_2} F \bullet \mathbf{dr} &= \int_0^{\sqrt[3]{2\pi}} (-\sin^2 t^3 - \cos^2 t^3 + 1/2\pi) 3t^2 dt = 1 - 2\pi \\
\int_{\Gamma_3} F \bullet \mathbf{dr} &= \int_0^{2\pi} (\sin^2 t + \cos^2 t + 1/2\pi) dt = 1 + 2\pi.
\end{aligned}$$

This shows that the line integral of F is path dependent.
Thus in view of FTLI the vector field F is not a gradient field.

Path independence

Definition: The integral $\int_\Gamma F \bullet \mathbf{dr}$ is said to be path independent if for any two paths Γ_1 and Γ_2 having the same initial and terminal points

$$\int_{\Gamma_1} F \bullet \mathbf{dr} = \int_{\Gamma_2} F \bullet \mathbf{dr}.$$

Theorem: Let F be a continuous vector field on U. Then

$$\int_\Gamma F \bullet \mathbf{dr} \text{ is path independent} \iff \int_\Gamma F \bullet \mathbf{dr} = 0$$

for every closed path Γ in U.

Proof: Consider $\Gamma = \Gamma_1 + \Gamma_2$ and $\Gamma = \Gamma_1 - \Gamma_2$.

An observation

Let F be a continuous vector field on an open set $U \subset \mathbb{R}^n$. Consider the following statements:

1. F is conservative on U.
2. $\int_\Gamma F \bullet \mathbf{dr}$ is path independent in U.
3. $\int_\Gamma F \bullet \mathbf{dr} = 0$ for any closed path in U.

By FLTI we have $(1) \Rightarrow (2) \Rightarrow (3)$. The implication $(3) \Rightarrow (1)$ holds under a suitable assumption of U.

Conservative vector fields and path independence

Definition: A subset $U \subset \mathbb{R}^n$ is said to be path connected if for any two points $\mathbf{x}$ and $\mathbf{y}$ in U there is a path $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(a) = \mathbf{x}, \gamma(b) = \mathbf{y}$ and $\gamma([a, b]) \subset U$.

Theorem-A: Let $U \subset \mathbb{R}^n$ be open and path connected and F be a continuous vector field on U. Suppose $\int_\Gamma F \bullet \mathbf{dr}$ depends only on the end points of Γ for any PC^1 path Γ in U. Then there exists a C^1 function $f : U \to \mathbb{R}$ such that $F = \nabla f$.

Further, for $\mathbf{a} \in U$, define $g : U \to \mathbb{R}$ by

$$g(\mathbf{x}) := \int_{\mathbf{a}}^{\mathbf{x}} F \bullet \mathbf{dr}$$

where the integral is taken over any PC^1 path joining $\mathbf{a}$ to $\mathbf{x}$. Then g is well defined, g is C^1 and $F = \nabla g$.

Conservative vector fields and path independence

Corollary: Let $U \subset \mathbb{R}^n$ be open and path connected and F be a continuous vector field on U. Then the following conditions are equivalent.

1. F is conservative on U, i.e., $F = \nabla f$ for some C^1 function $f : U \to \mathbb{R}$.
2. $\int_\Gamma F \bullet \mathbf{dr}$ is path independent for any PC^1 path in U.
3. $\int_\Gamma F \bullet \mathbf{dr} = 0$ for any PC^1 closed path in U.

Proof of Theorem-A

By hypothesis $g(\mathbf{x}) := \int_{\mathbf{a}}^{\mathbf{x}} F \bullet \mathbf{dr}$ is well defined.

1. $g(\mathbf{x} + h\mathbf{e}_i) - g(\mathbf{x}) = \displaystyle\int_{\mathbf{x}}^{\mathbf{x}+h\mathbf{e}_i} F \bullet \mathbf{dr}.$
2. Consider $\mathbf{r}(t) = \mathbf{x} + th\mathbf{e}_i,\ t \in [0, 1]$. Then $\mathbf{dr} = h\mathbf{e}_i dt$ and
$$\frac{g(\mathbf{x} + h\mathbf{e}_i) - g(\mathbf{x})}{h} = \int_0^1 F(\mathbf{x} + th\mathbf{e}_i) \bullet \mathbf{e}_i dt.$$
3. Setting $u = th \Rightarrow du = hdt$. Hence
$$\int_0^1 F(\mathbf{x} + th\mathbf{e}_i) \bullet \mathbf{e}_i dt = \frac{1}{h}\int_0^h F_i(\mathbf{x} + u\mathbf{e}_i) du \to F_i(\mathbf{x}).$$

Exact differentials

Let F be a vector field on U with a scalar potential f, that is, $F = \nabla f$. Suppose $F = (F_1, \ldots, F_n)$. Then the differential

$$F \bullet \mathbf{dr} = F_1 dx_1 + \cdots + F_n dx_n$$

is called exact.

Fact: If a C^1 vector field $F = (F_1, \ldots, F_n)$ on U is conservative then for all i and j

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

Proof: We have $F_i = \partial_i f \Rightarrow \partial_j F_i = \partial_j \partial_i f = \partial_i \partial_j f = \partial_i F_j$.

Example

Consider $F(x, y) := (3 + 2xy,\, x^2 - 3y^2) =: (P, Q)$. Then $Q_x = 2x = P_y$ so the necessary condition is satisfied.

We wish to find f such that $F = \nabla f$. If f exists then $f_x(x, y) = 3 + 2xy \Rightarrow f(x, y) = 3x + x^2y + h(y)$.

Thus $f_y(x, y) = x^2 + h'(y) = x^2 - 3y^2 \Rightarrow h'(y) = -3y^2$. Hence $h(y) = -y^3 + c$ for some constant c. Consequently,

$$f(x, y) = 3x + x^2y - y^3 + c \text{ and } F = \nabla f.$$

Example

Consider $F(x, y) := (\frac{-y}{x^2 + y^2}, \ \frac{x}{x^2 + y^2}) = (P, Q)$ for $(x, y) \neq (0, 0)$. Then we have $Q_x = P_y$ so the necessary condition is satisfied.

For the path $\Gamma : \mathbf{r}(t) = (\cos t, \sin t), \ t \in [0, 2\pi]$, we have

$$\int_\Gamma F \bullet \mathbf{dr} = \int_0^{2\pi} dt = 2\pi.$$

This shows that F is not conservative.

Remark: The necessary condition $\partial_i F_j = \partial_j F_i$ is also sufficient for conservativeness of F when the domain of F is simply connected. This is a consequence of Green's theorem.

*** End ***