Lecture 11:
Arclength and Line Integrals

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Parametric curves

Definition:

- A continuous mapping $\gamma : [a, b] \to \mathbb{R}^n$ is called a parametric curve or a parametrized path and $[a, b]$ is called the parameter space.
- The set $\Gamma := \gamma([a, b])$ is called a geometric curve or a geometric path in $\mathbb{R}^n$.

Examples:

- The parametric path $\gamma(t) := (\cos t, \sin t)$ for $t \in [0, 2\pi]$ is a circle in $\mathbb{R}^2$.
- The parametric path $\gamma(t) := (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$ is helix in $\mathbb{R}^3$.

Polygonal approximations of paths

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parametric path. For a partition $P := \{t_0, \ldots, t_m\}$ of $[a, b]$, define

$$\ell(P, \gamma) := \sum_{j=1}^{m} \|\gamma(t_j) - \gamma(t_{j-1})\|$$

and $\mu(P) := \max\{t_j - t_{j-1} : j = 1 : m\}$.

Note that $\ell(P, \gamma) \leq \ell(Q, \gamma)$ if Q is a refinement of P. Hence

$$\lim_{\mu(P) \to 0} \ell(P, \gamma) = \sup_{P} \ell(P, \gamma).$$

Arclength of a curve

Definition: Let $\ell(\gamma) := \sup_P \ell(P, \gamma)$. If $\ell(\gamma)$ is finite then γ is said to be rectifiable (finite length) and $\ell(\gamma)$ is said to be the arclength of γ.

Theorem: Let $\gamma : [a.b] \to \mathbb{R}^n$ be a C^1 (or PC^1) path. Then γ is rectifiable and

$$\ell(\gamma) = \lim_{\mu(P)\to 0} \ell(P, \gamma) = \int_a^b \|\gamma'(t)\| dt.$$

Proof: Use $\gamma(t_j) - \gamma(t_{j-1}) = \gamma'(t_{j-1})\Delta t_j + e(\Delta t_j)\Delta t_j$ with $e(\Delta t_j) \to 0$ as $\Delta t_j \to 0$ and the Riemann sum of $\|\gamma'(t)\|$.

Arclength

- If $f : [a, b] \to \mathbb{R}$ is C^1 then the length of the graph of f is given by
$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

- If $\gamma : [a, b] \to \mathbb{R}^2$ is C^1 and $\gamma(t) = (x(t), y(t))$ then
$$\ell(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

- If $\gamma : [a, b] \to \mathbb{R}^3$ is C^1 and $\gamma(t) = (x(t), y(t)), z(t))$ then
$$\ell(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Examples

- The arclength of the helix $\gamma(t) := (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$ is given by

$$\int_0^{2\pi} \|\gamma'(t)\| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\,\pi.$$

- The arclength of $\gamma(t) := (\cos t, \sin t, \cos(2t), \sin(2t))$ for $t \in [0, \pi]$ is given by

$$\int_0^{\pi} \|\gamma'(t)\| dt = \int_0^{2\pi} \sqrt{5} dt = \sqrt{5}\,\pi.$$

Invariance of arclength

Obviously the arclength $\ell(\gamma)$ depends to the parametrization γ. However, $\ell(\gamma)$ is invariant under equivalent parametrization.

Definition: Two parametric curves $\gamma : [a, b] \to \mathbb{R}^n$ and $\rho : [c, d] \to \mathbb{R}^n$ are said to be equivalent if there exits a continuous function $u : [a, b] \to [c, d]$ such that

- u is strictly increasing
- $u([a, b]) = [c, d]$ and
- $\gamma = \rho \circ u$.

Theorem: Let $\gamma : [a, b] \to \mathbb{R}^n$ be rectifiable. Then every parametric curve equivalent to γ is rectifiable and has the same arclength $\ell(\gamma)$.

Arclength differential

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1 path. Define $s : [a, b] \to [0, \ell]$ by

$$s(t) := \int_a^t \|\gamma'(\tau)\| \tau,$$

where $\ell := \ell(\gamma)$. Then

$$\frac{ds}{dt} = \|\gamma'(t)\| \quad \text{or equivalently} \quad ds = \|\gamma'(t)\| dt.$$

- ds is called the arclength differential and is written as

$$ds = \sqrt{dx^2 + dy^2} \text{ when } \gamma(t) = (x(t), y(t))$$
$$ds = \sqrt{dx^2 + dy^2 + dz^2} \text{ when } \gamma(t) = (x(t), y(t), z(t))$$

Smooth parametrization

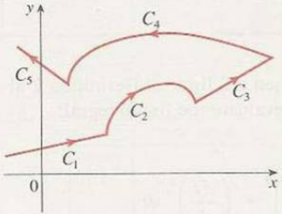

FIGURE 4
A piecewise-smooth curve

A parametric curve $\Gamma : [a, b] \to \mathbb{R}^n$ is said to be

- smooth if γ is C^1 on $[a, b]$ and $\gamma'(t) \neq 0$ for $t \in (a, b)$,
- piecewise smooth if γ is smooth on $[t_{j-1}, t_j]$ for some partition $t_0, \ldots, t_m$ of $[a, b]$.

Smooth parametrization by arclength

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth (or piecewise smooth) curve. Then

$$\frac{ds}{dt} = \|\gamma'(t)\| > 0 \Longrightarrow s(t) = \int_a^t \|\gamma'(t)\| dt$$

is strictly increasing and $s([a, b]) = [0, \ell]$. Consequently

$$\rho : [0, \ell] \to \mathbb{R}^n \text{ given by } \rho(u) = \gamma(s^{-1}(u))$$

is equivalent to γ. Moreover, $\|\rho'(u)\| = 1$ which gives

$$\ell(\rho) = \int_0^\ell \|\rho'(u)\| du = \ell = \ell(\gamma).$$

Examples

- Consider the helix $\gamma(t) := (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$. Then
$$s(t) = \int_0^t \sqrt{2}\, dt = \sqrt{2}t \Rightarrow t = s/\sqrt{2}.$$
Hence $\rho : [0,\ 2\sqrt{2}\pi] \to \mathbb{R}^3$ given by
$$\rho(s) := (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}),\ s/\sqrt{2})$$
is a smooth parametrization of the helix γ by arclength.
- Consider $\gamma(t) := (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Then
$$s = \int_0^t dt = t \Rightarrow \rho(s) := (\cos(s),\ \sin(s)),\ \ s \in [0, 2\pi]$$
is a smooth parametrization of γ by arclength.

Partition of curves

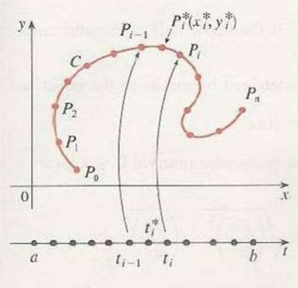

FIGURE 1
A partition of $[a,b]$ determines a partition of C

Let Γ be a curve in $\mathbb{R}^n$ paramatrized by $\mathbf{r} : [a, b] \to \mathbb{R}^n$. Then a partion $P := (a = t_0 < \ldots < t_m = b)$ of $[a, b]$ induces a partition of Γ into m subarcs with arclenths $\Delta s_1, \ldots, \Delta s_m$.

$$\text{Define} \quad \mu(P) := \max_{1 \le j \le m} \Delta s_j.$$

Riemann sum of scalar field w.r.t. arclength

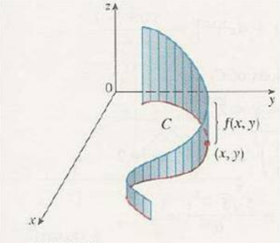

FIGURE 2

Let $f : \Gamma \to \mathbb{R}$. Then for any $\mathbf{p}_j$ in the j-th subarc, consider the Riemann sum of f w.r.t. to the arclength

$$S(P, f) := \sum_{j=1}^{m} f(\mathbf{p}_j)\Delta s_j.$$

Line integrals of scalar fields w.r.t. arclength

Definition: Suppose that $\mathbf{r}$ is PC^1 and $f : \Gamma \to \mathbb{R}$. Then the line integral of f along Γ w.r.t. the arclength is given by

$$\int_\Gamma f(\mathbf{x})ds := \lim_{\mu(P)\to 0} S(P, f) = \lim_{\mu(P)\to 0} \sum_{j=1}^{m} f(\mathbf{p}_j)\Delta s_j$$

if the limit exists.

Fact: If f is continuous and $\mathbf{r}(t)$ is PC^1 then we have

$$\int_\Gamma f(\mathbf{x})ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt.$$

Proof: Since $\Delta s \simeq \|\mathbf{r}'(t)\|\Delta t$, i.e., $ds = \|\mathbf{r}'(t)\|dt$ and $t \mapsto f(\mathbf{r}(t))\|\mathbf{r}'(t)\|$ is piecewise continuous, the result follows.

Line integrals of scalar fields

For the plane curve $\Gamma : \mathbf{r}(t) = (x(t), y(t)), t \in [a, b]$ we have

$$\int_\Gamma f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\; dt.$$

Example: Evaluate $\int_\Gamma (2 + x^2 y)ds$, where Γ is the upper half of the circle $x^2 + y^2 = 1$.

Considering $x(t) = \cos t, y(t) = \sin t, 0 \leq t \leq \pi$, we have

$$\int_\Gamma (2 + x^2 y)ds = \int_0^\pi (2 + \cos^2 t \sin t)dt = 2\pi + 2/3.$$

Application of line integrals

Suppose a thin wire in the shape of a curve Γ parametrized by a C^1 path $\mathbf{r} : [a, b] \to \mathbb{R}^2$ has density $\rho(x, y)$. Then the total mass of the wire is given by

$$m = \int_\Gamma f(x, y)ds.$$

The center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{1}{m}\int_\Gamma x\rho(x, y)ds, \quad \bar{y} = \frac{1}{m}\int_\Gamma y\rho(x, y)ds.$$

The moment of inertia about a line L is given by

$$I_L = \int_\Gamma \delta(x, y)^2\rho(x, y)ds,$$

where $\delta(x, y)$ is shortest distance from (x, y) to L.

Example

The wire W has the shape $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is parametrized by $\gamma_1(t) := (\cos t, \sin t)$, $t \in [0, \pi]$ and Γ_2 is parametrized by $\gamma_2(t) := (t, 0)$, $t \in [-1, 1]$. The density is given by $f(x, y) := \sqrt{x^2 + y^2}$.

Then $\|\gamma_i'(t)\| = 1$, $f(\gamma_1(t)) = 1$ and $f(\Gamma_2(t)) = |t|$. Thus

$$
\begin{aligned}
m &= \int_\Gamma f ds = \int_0^\pi dt - \int_{-1}^0 t dt + \int_0^1 t dt = \pi + 1, \\
\bar{x} &= \frac{1}{\pi + 1}\left(\int_0^\pi \cos t dt - \int_{-1}^0 t^2 dt + \int_0^1 t^2 dt\right) = 0, \\
\bar{y} &= \frac{1}{\pi + 1}\left(\int_0^\pi \sin t dt + 0\right) = \frac{2}{\pi + 1}.
\end{aligned}
$$

Example (contd.)

For the moment of inertia I_x about x axis, we have $\delta(x, y) = |y|$ and hence

$$I_x = \int_0^{\pi} \sin^2 t dt + 0 = \int_0^{\pi} \frac{1}{2}(1 - \cos(2t))dt = \frac{\pi}{2}.$$

For the moment of inertia I_y about y axis, we have $\delta(x, y) = |x|$ and hence

$$I_y = \int_0^{\pi} \cos^2 t dt - \int_{-1}^{0} t^3 dt + \int_0^{1} t^3 dt = \frac{\pi}{2} + \frac{1}{2}.$$

Properties of line integrals of scalar fields

Fact: Let Γ be parametrized by a PC^1 curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ and $f, g : \Gamma \to \mathbb{R}$ be continuous. Then the following hold:

- $\int_\Gamma fds$ is invariant under equivalent parametrization of Γ.
- $\int_\Gamma (f + \alpha g)ds = \int_\Gamma fds + \alpha \int_\Gamma gds$ for $\alpha \in \mathbb{R}$.
- Let $\Gamma = \Gamma_1 + \cdots + \Gamma_m$, where Γ_i is parametrized by C^1 curve $\mathbf{r}_i : [a_i, b_i] \to \mathbb{R}^n$. Then

$$\int_\Gamma fds = \int_{\Gamma_1} fds + \cdots + \int_{\Gamma_m} fds.$$

Example

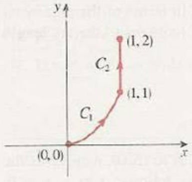

FIGURE 5
$C = C_1 \cup C_2$

Evaluate $\int_\Gamma 2x ds$, where Γ consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment C_2 from $(1,1)$ to $(1,2)$. Then

$$\int_\Gamma 2x ds = \int_{C_1} 2x ds + \int_{C_2} 2x ds = \frac{1}{6}(5\sqrt{5} + 11).$$

*** End ***