MA 102 (Multivariable Calculus)

Quiz-2

Date: March 28, 2013 **Time:** 50 minutes **Maximum Marks:** 20

Answer ALL questions

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^1. Show that

$$N = \frac{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

is a unit normal vector to the graph of f. **2 marks**

Solution: Consider $F(x, y, z) = f(x, y) - z$. Then $\nabla F = (\nabla f, -1)$ is normal to the surface $F(x, y, z) = 0$. **1 mark**

This shows that $(f_x, f_y, -1)$ is normal to the surface $z = f(x, y)$. Normalizing $(f_x, f_y - 1)$ we obtain N. **1 mark**

2. Find the normal line and the tangent plane to the surface $z = xe^y$ at the point $(1, 0, 1)$. **4 marks**

Solution: Consider $F(x, y, z) := xe^y - z$. Then $\nabla F = (e^y, xe^y, -1)$. Therefore $\nabla F(1, 0, 1) = (1, 1, -1)$. The equation of the tangent to the surface $F(x, y, z) = 0$ at $(1, 0, 1)$ is given by

$$[(x, y, z) - (1, 0, 1)] \bullet \nabla F(1, 0, 1) = 0,$$

which gives $x - 1 + y - (z - 1) = 0 \Rightarrow z = x + y$. **2 marks**

The equation of the normal line to the surface $F(x, y, z) = 0$ at $(1, 0, 1)$ is given by

$$(x, y, z) = (1, 0, 1) + t\nabla F(1, 0, 1) = (1, 0, 1) + t(1, 1, -1).$$

2 marks

The line can also be written as $\dfrac{x-1}{1} = \dfrac{y-0}{1} = \dfrac{z-1}{-1}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := \dfrac{x^2 y}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that f is continuous at $(0, 0)$. Show that the directional derivative $\mathrm{D}_u f(0, 0)$ exists for all $u \in \mathbb{R}^2$ and determine $\mathrm{D}_u f(0, 0)$. Is f differentiable at $(0, 0)$? **6 marks**

Solution: We have $|f(x, y) - f(0, 0)| = |x^2 y|/(x^2 + y^2) \leq |x|(x^2 + y^2)/(x^2 + y^2) = |x| \to 0$ as $(x, y) \to 0$. Hence f is continuous. **2 marks**

Let u be a unit vector, that is, $\|u\| = 1$. Then $D_u f(0, 0) = \lim_{t\to 0} \frac{f(0+tu) - f(0)}{t} = \lim_{t\to 0} u_1^2 u_2 = u_1^2 u_2$. Hence $D_u f(0, 0)$ exists for all nonzero unit u and $D_u f(0, 0) = u_1^2 u_2$. **2 marks**

We have $f_x(0, 0) = 0 = f_y(0, 0)$. Hence $D_u f(0, 0) \neq \nabla f(0, 0) \bullet u$. Therefore f is not differentiable. **2 marks**

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where m is a nonnegative integer. If f is differentiable then show that $\langle x, \nabla f(x) \rangle = m f(x)$. **2 marks**

   Solution: Set $\phi(t) = f(tx)$. Then by composition rule ϕ is differentiable. Hence chain rule $\phi'(t) = \nabla f(tx) \bullet x$. **1 mark**

   On the other hand, $\phi'(t) = mt^{m-1} f(x)$. Hence we have $\nabla f(tx) \bullet x = \phi'(t) = mt^{m-1} f(x)$. Taking $t = 1$ the result follows. $\blacksquare$. **1 mark**

5. Find maxima, minima and saddle point, if any, of the function $f(x, y) := 4xy - 2x^2 - y^4$

   6 marks

   Solution: Solving $f_x = 4y - 4x = 0$ and $f_y = 4x - 4y^3 = 0$ we obtain the critical points $(0, 0), (1, 1)$ and $(-1, -1)$. **2 marks**

   We have $f_{xx} = -4, f_{yy} = -12y^2$ and $f_{xy} = 4$. This shows that $D = f_{xx} f_{yy} - f_{xy}^2 = 48y^2 - 16$. **1 marks**

   At $(0, 0)$ $D < 0$ therefore $(0, 0)$ is a saddle point. **1 mark**

   At $(1, 1)$ and $(-1, -1)$ we have $f_{xx} < 0$ and $D > 0$. Hence f has local maximum there.

   1 mark.

   The maximum value of f is given by $f(1, 1) = f(-1, -1) = 1$. **1 mark**

**** End ****