1. Consider the one parameter family of curves $x^2 = ce^y$ where c is an arbitrary constant. The trajectory through the point $x = 0, y = 0$ which is orthogonal to the given family also passes through the point $(x = 2, y = --)$. [2]

Answer:

$$x^2 = ce^y \implies ce^y \frac{dy}{dx} = 2x \implies \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}.$$

The orthogonal trajectory is given by the ODE

$$\frac{dy}{dx} = \frac{-x}{2}, \qquad y(0) = 0.$$

Hence, the orhtogonal trajectory is $y = -\frac{x^2}{4}$ and $y(2) = -1$.

2. (a) Determine three linearly independent solutions of the homogeneous system [3]

$\frac{dx}{dt} = x + y$

$\frac{dy}{dt} = -x + 3y$

$\frac{dz}{dt} = -x + y + 2z.$

Answer: The coefficient matrix A has eigenvalues $2, 2, 2$. [0.5]

Two linearly independent eigenvectors can be taken as $\overrightarrow{v} = [1\ \ 1\ \ 0]^T$ and $\overrightarrow{w} = [0\ \ 0\ \ 1]^T$. [0.5]

Thus we get two linearly independent solutions $e^{2t}[1\ \ 1\ \ 0]^T$ and $e^{2t}[0\ \ 0\ \ 1]^T$. [0.5]

The third linealy independent solution is of the form $e^{2t}(t\overrightarrow{a} + \overrightarrow{b})$ where $\overrightarrow{a} = k_1\overrightarrow{v} + k_2\overrightarrow{w}$ gives a consistent system of linar equations for $(A - 2I)\overrightarrow{b} = \overrightarrow{a}$. [0.5]

Solving, we find that $\overrightarrow{a} = [1\ \ 1\ \ 1]^T$ and $\overrightarrow{b} = [0\ \ 1\ \ 0]^T$. [1]

Remark: There may be other valid choices for $\overrightarrow{b}$, for example, $\overrightarrow{b} = [1\ \ 2\ \ 0]^T$ or $[-1\ \ 0\ \ 0]^T$.

(b) Find out the critical solution(s) and discuss stability of each critical solution. [2]

Answer: Solving $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ and $\frac{dz}{dt} = 0$, we get $(0, 0, 0)$ as the only crtical solution. [1]

As the eigenvalues are positve, the critical solution is unstable. [1]

(c) Find the solution of the IVP consisting of the above system together with the initial condition $x(0) = 2$, $y(0) = 3$, $z(0) = 4$. [2]

Answer: We need to find c_1, c_2 and c_3 such that for $t = 0$,

$$c_1e^{2t}[1\ \ 1\ \ 0]^T + c_2e^{2t}[0\ \ 0\ \ 1]^T + c_1e^{2t}\big(t[1\ \ 1\ \ 1]^T + [0\ \ 1\ \ 0]^T\big) = [2\ \ 3\ \ 4]^T \qquad (t = 0). \quad [1]$$

We get $c_1 = 2$, $c_2 = 4$ and $c_3 = 1$ [1]

(**or equivalently** $x(t) = e^{2t}(t + 2)$, $y(t) = e^{2t}(t + 3)$, $z(t) = e^{2t}(t + 4)$. [1])