MA 102 (Mathematics II)

End Semester Examination

Date: April 23, 2013 **Time:** 3 Hours **Maximum Marks:** 50

Question 4:

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 scalar field such that $f(x, y, z) \neq 0$ for $(x, y, z) \in \mathbb{R}^3$, $\|\nabla f\|^2 = 4f$ and $\operatorname{div}(f\nabla f) = 10f$. Evaluate $\iint_S \nabla f \bullet \mathbf{n} dS$, where S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. [3]

Solution: We have $\operatorname{div}(f\nabla f) = \|\nabla f\|^2 + f\operatorname{div}(\nabla f)$. **[1/2 mark]**

This gives $f\operatorname{div}(\nabla f) = 10f - 4f = 6f \Rightarrow \operatorname{div}(\nabla f) = 6$ as $f \neq 0$. **[1 mark]**

Now by Divergence theorem,

$$\begin{aligned} \iint_S \nabla f \bullet \mathbf{n} dS &= \iiint_V \operatorname{div}(\nabla f) dV && [\mathbf{1/2\ mark}] \\ &= 6 \iiint dV = 6 \times \frac{4\pi}{3}. && [\mathbf{1\ mark}] \end{aligned}$$

(b) Let V be the solid region within the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 4$ and the paraboloid $z = 1 - x^2 - y^2$. Evaluate $\iiint_V \sqrt{x^2 + y^2} dV$. [2]

Solution: Considering cylindrical coordinates, we have

$$V = \{(r, \theta, z) : r \in [0, 1], \theta \in [0, 2\pi], 1 - r^2 \leq z \leq 4\}. \qquad [\mathbf{1mark}]$$

Hence

$$\iiint_V \sqrt{x^2 + y^2} dV = \int_0^1 \int_0^{2\pi} \int_{1-r^2}^4 r^2 dr d\theta dz = \frac{12\pi}{5}. \qquad [\mathbf{1mark}]$$

(c) Use Divergence theorem to evaluate $\iint_S F \bullet \mathbf{n} dS$, where $F = (xy, y^2 + e^{xz^2}, \sin(xy))$ and S is the boundary of the solid region V bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0, y = 0$ and $y + z = 2$. [3]

Solution:

We have $V = \{(x, y, z) : -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2, 0 \leq y \leq 2 - z\}$. **[1 mark]**

By Divergence theorem (**Zero mark if Divergence theorem is not used**)

$$\begin{aligned} \iint_S F \bullet d\mathbf{S} &= \iiint_V \operatorname{div}(F) dV = \iiint_V 3y dV && [\mathbf{1mark}] \\ &= 3 \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} y dy dz dx = \frac{184}{35}. && [\mathbf{1mark}] \end{aligned}$$

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