MA 102 (Mathematics II)

End Semester Examination

Date: April 23, 2013 **Time:** 3 Hours **Maximum Marks:** 50

Question 3:

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) := \sin(y^2/x) \cdot \sqrt{x^2+y^2}$ if $x \neq 0$ and $f(x,y) = 0$ if $x = 0$. Show that f is continuous at $(0,0)$ and has directional derivatives in every direction at $(0,0)$. Is f differentiable at $(0,0)$? [3]

Solution: We have $|f(x,y) - f(0,0)| = |\sin(y^2/x)|\sqrt{x^2+y^2} \leq \sqrt{x^2+y^2} \to 0$ as $\|(x,y)\| = \sqrt{x^2+y^2} \to 0$. Hence f is continuous at $(0,0)$. **[1 mark]**

Let $u = (u_1, u_2)$ be a unit vector. If $u_1 u_2 = 0$ then it follows that

$$D_u f(0,0) = \lim_{t\to 0} \frac{f(tu) - f(0,0)}{t} = 0.$$

Now suppose that $u_1 u_2 \neq 0$. Then

$$D_u f(0,0) = \lim_{t\to 0} \frac{f(tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{\sin(tu_2^2/u_1)|t|}{t} = 0.$$

Thus $D_u f(0,0)$ exists for all unit vector u. **[1 mark]**

However, f is not differentiable. Indeed, we have

$$\frac{|f(h,k) - f(0,0) - (f_x(0,0)h + f_y(0,0)k)|}{\sqrt{h^2+k^2}} = |\sin(k^2/h)| \to |\sin(1/m)| \not\to 0$$

as $(h,k) \to (0,0)$ along the path $h = mk^2$. **[1 mark]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $g : \mathbb{R}^n \to \mathbb{R}$ be given by $g(\mathbf{x}) := f(\|\mathbf{x}\|^2)$. Show that g is differentiable and $\|\nabla g(\mathbf{x})\|^2 = 4\|\mathbf{x}\|^2(f'(\|\mathbf{x}\|^2))^2$. [2]

Solution: Let $h(x) = \|x\|^2 = x_1^2 + \cdots + x_n^2$. Then $\partial_{x_j} h(x) = 2x_j$ and hence $\nabla h(x) = 2x$. Since the partial derivatives are continuous, h is differentiable. **[1 mark]**

As $g = f \circ h$, by chain rule g is differentiable and $\nabla g(x) = f'(h(x))\nabla h(x) = 2xf'(\|x\|^2)$. This shows that $\|\nabla g(\mathbf{x})\|^2 = 4\|\mathbf{x}\|^2(f'(\|\mathbf{x}\|^2))^2$. **[1 mark]**

(c) Find the equation of the tangent plane and the normal line to the graph of $f(x,y) := x^2 - y^4 + e^{xy}$ at the point $(1,0,2)$ [2]

Solution: The tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and the normal line is given by $(x, y, z) = (x_0, y_0, z_0) + t(f_x, f_y, -1)$ or equivalently

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{-1}.$$

We have $f_x = 2x + ye^{xy}$ and $f_y = 4y^3 + xe^{xy} \Rightarrow f_x(1, 0, 2) = 2$ and $f_y(1, 0, 2) = 1$. Hence the equation of the tangent plane at $(1, 0, 2)$ is given by

$$z = 2(x - 1) + 1(y - 0) + 2 \Rightarrow z = 2x + y. \qquad \textbf{[1 mark]}$$

The equation of the normal line is $(x, y, z) = (1, 0, 2) + t(2, 1, -1)$ or equivalently

$$\frac{x - 1}{2} = \frac{y}{1} = \frac{z - 2}{-1}. \qquad \textbf{[1 mark]}$$

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