Echelon forms and linear systems

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Outline

- Gaussian elimination
- Echelon form (Ref)
- Gauss-Jordan elimination and Rref

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- Rank of a matrix
- Elementary matrices

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- Elementary matrices
- Invertible matrices
- LU Factorization

Gaussian elimination (GE)

Linear system $\longrightarrow$ Upper triangular system $\longrightarrow$ Solution

$$Ax = b \longrightarrow Ux = d \Longrightarrow x$$

$$\left[A \quad | \quad b\right] \longrightarrow \left[U \quad | \quad d\right] \Longrightarrow x$$

Gaussian elimination (GE)

Linear system $\longrightarrow$ Upper triangular system $\longrightarrow$ Solution

$$Ax = b \longrightarrow Ux = d \Longrightarrow x$$

$$\begin{bmatrix} A & | & b \end{bmatrix} \longrightarrow \begin{bmatrix} U & | & d \end{bmatrix} \Longrightarrow x$$

Tools of the Trade: Elementary row operations

- Multiply a row by nonzero scalar: $\mathrm{row}_i(A) \longrightarrow \alpha\, \mathrm{row}_i(A)$.
- Add a row to another row: $\mathrm{row}_i(A) + \mathrm{row}_j(A) \longrightarrow \mathrm{row}_j(A)$.
- Row exchange: $\mathrm{row}_i(A) \leftrightarrow \mathrm{row}_j(A)$

Exercise: Describe inverse operations.

Forward elimination (Forward GE)

Square matrix:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \to \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \to \begin{bmatrix} p_{11} & * & * \\ 0 & p_{22} & * \\ 0 & 0 & p_{33} \end{bmatrix}.$$

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Rectangular matrix:

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \to \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \to \begin{bmatrix} p_{11} & * & * & * \\ 0 & p_{22} & * & * \\ 0 & 0 & p_{33} & * \end{bmatrix}$$

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Example 1

$$\textbf{Square system:}\quad \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Forward elimination (Forward GE) $\longrightarrow$ Upper triangular form:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ -1 & 2 & 0 & 1 \\ 1 & -1 & 2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ -1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right]$$

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$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & -3 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -9 & 9 \end{array}\right]$$

Back substitution: $x_3 = -1, x_2 = 1$ and $x_1 = 1$.

Example 2

$$\textbf{Square System:} \quad \begin{bmatrix} 0 & 1 & 5 \\ 1 & 4 & 3 \\ 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}$$

Forward GE:

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{array}\right]$$

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$$\to \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{array}\right] \Longrightarrow \textbf{No solution}$$

Example 3

$$\textbf{Nonsquare system:} \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Forward GE:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & -1 & 2 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 2 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & -3 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Back substitution: $x_3 = t, x_2 = 2 + t$ and $x_1 = -t$ for $t \in \mathbb{R}$.

Echelon form

Pivot: First nonzero entry in a row is called a pivot (leading entry).

Pivot column: A column containing a pivot (leading entry) is called a pivot column.

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Echelon form: An $m \times n$ matrix A is in echelon form provided:

- All zero rows appear at the bottom.
- The pivot (leading entry) in a row is always to the right of the pivot of the row above it.

Notation: $\mathrm{Ref}(A) =$ row echelon form of A.

Examples

Matrices in echelon form:

$$\begin{bmatrix} p & * & * & * & * & * & * & * \\ 0 & 0 & p & * & * & * & * & * \\ 0 & 0 & 0 & p & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & p & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} p & * & * & * & * \\ 0 & 0 & p & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} p & * & * \\ 0 & p & * \\ 0 & 0 & p \end{bmatrix}.$$

Matrices not in echelon form:

$$\begin{bmatrix} 2 & 3 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Forward GE and echelon form

Forward GE:

$$m \times n \text{ matrix } A \longrightarrow \text{ Upper triangular form } U.$$

Forward GE with additional restrictions on pivot entries:

$$m \times n \text{ matrix } A \longrightarrow \text{ echelon form } \mathrm{Ref}(A).$$

Remark: Echelon form of A is NOT unique.

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Echelon form via forward GE:

$$\begin{bmatrix} 0 & 1 & 5 \\ 1 & 4 & 3 \\ 2 & 7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 5 \\ 2 & 7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Echelon form and consistency

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Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{array}\right] = \text{echelon form} \Rightarrow \text{ inconsistent}$$

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An $m \times n$ matrix A is in reduced row echelon form provided:

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An $m \times n$ matrix A is in reduced row echelon form provided:

- A is in row echelon form.
- Each pivot (leading entry) in A is 1.
- Pivot is the only nonzero entry in a pivot column.

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Matrices in echelon form:

$$\begin{bmatrix} 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Gauss-Jordan elimination and rref

Forward GE : $m \times n$ matrix $A \longrightarrow \mathrm{Ref}(A)$.
Backward GE: $\mathrm{Ref}(A) \longrightarrow \mathrm{Rref}(A)$.

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Gauss-Jordan elimination $=$ Forward GE followed by backward GE.

Gauss-Jordan elimination: $m \times n$ matrix $A \longrightarrow \mathrm{Rref}(A)$.

Theorem: Reduced row echelon form of an $m \times n$ matrix A is unique.

Example: Gauss-Jordan elimination

Forward GE: $A \to \mathrm{Ref}(A)$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

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Backward GE: $\mathrm{Ref}(A) \to \mathrm{Rref}(A)$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Rank of a matrix

Rank: The rank of an $m \times n$ matrix A, denoted by $\mathrm{rank}(A)$, is the number of pivots in $\mathrm{Rref}(A)$.

$$A := \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{rank}(A) = 2.$$

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Fact:

- $\mathrm{rank}(A)$ = number of pivot columns in $\mathrm{Rref}(A)$ = number of nonzero rows in $\mathrm{Rref}(A)$.
- $\mathrm{rank}(A)$ = number of pivot columns in $\mathrm{Ref}(A)$ = number of nonzero rows in $\mathrm{Ref}(A)$.

Leading and free variable:

Free variable: A variable in a system $Ax = b$ is called a free variable if the system has a solution for every value of that variable.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 3 \\ 2 & 1 & 4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -1 - 3x_3 \\ x_2 = 3 + 2x_3 \\ x_3 : \ \text{free} \end{array}$$

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Leading variables: Let $[A \mid b] \longrightarrow \mathrm{Rref}([A \mid b]) =: [R \mid d]$. Then the variables corresponding to the pivot columns of R are called leading variable.

Theorem: The number of free variables in a consistent $m \times n$ system $Ax = b$ is given by $n - \mathrm{rank}(A)$.

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Theorem: The number of free variables in a consistent $m \times n$ system $Ax = b$ is given by $n - \mathrm{rank}(A)$.

Proof: $\#$ Free variables $= \#$ non-pivot columns $= n - \mathrm{rank}(A)$.

Rank and consistency

Fact: An $m \times n$ homogeneous system $Ax = 0$ has

- infinitely many solutions if $\mathrm{rank}(A) < n$,
- unique (trivial) solution if $\mathrm{rank}(A) = n$.

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- has unique solution if $\operatorname{rank}(A) = \operatorname{rank}([A \mid b]) = n$.
- infinitely many solutions if $\operatorname{rank}(A) = \operatorname{rank}([A \mid b]) < n$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 2 & k \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & k-1 \end{array}\right] \Rightarrow \text{ inconsistent if } k \neq 1.$$

Elementary matrices

Type	Operation	Inverse operation
I	$R_i \longrightarrow \alpha\, R_i$	$R_i \longrightarrow \frac{1}{\alpha}\, R_i$
II	$cR_i + R_j \longrightarrow R_j$	$-cR_i + R_j \longrightarrow R_j$
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An elementary matrix is a matrix that is obtained by performing an elementary row operation on the identity matrix.

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$$\text{Type I}: E_2(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha x_2 \\ x_3 \end{bmatrix}.$$

$$E_2(\alpha)A = \begin{bmatrix} \mathrm{row}_1(A) \\ \alpha\,\mathrm{row}_2(A) \\ \mathrm{row}_3(A) \end{bmatrix} = \text{ multiply 2nd row of } A \text{ by } \alpha.$$

$$(E_2(\alpha))^{-1} = E_2(\frac{1}{\alpha}).$$

Type II elementary matrices

$$\text{Type II}: E_{13}(2) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = I_3 + 2e_3 e_1^T$$

The matrix E_{13} is obtained by performing $2R_1 + R_3 \to R_3$ on I_3.

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$$E_{13}(2)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + 2x_1 \end{bmatrix} \Rightarrow E_{13}A = \begin{bmatrix} \mathrm{row}_1(A) \\ \mathrm{row}_2(A) \\ \mathrm{row}_3(A) + 2\,\mathrm{row}_1(A) \end{bmatrix}.$$

$(E_{13}(2))^{-1} = E_{13}(-2)$ corresponds to $-2R_1 + R_3 \to R_3$ on I_3.

Type III elementary matrices

Type III : E_{ij} is obtained by performing $R_i \leftrightarrow R_j$ on I.

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow E_{23} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} \Rightarrow E_{23}A = \begin{bmatrix} \mathrm{row}_1(A) \\ \mathrm{row}_3(A) \\ \mathrm{row}_2(A) \end{bmatrix}.$$

Type III elementary matrices

Type III : E_{ij} is obtained by performing $R_i \leftrightarrow R_j$ on I.

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow E_{23} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} \Rightarrow E_{23}A = \begin{bmatrix} \mathrm{row}_1(A) \\ \mathrm{row}_3(A) \\ \mathrm{row}_2(A) \end{bmatrix}.$$

$$(E_{ij})^{-1} = E_{ij} \text{ corresponds to row operation } R_i \leftrightarrow R_j \text{ on } I.$$

Observation: Inverse of an elementary matrix is also an elementary matrix of same type.

Row operation via elementary matrices

Crux of the matter:

- Type I: Multiplying $E_i(c)$ to A giving $E_i(c)A$ amounts to performing the row operation $cR_i \to R_i$ on A.
- Type II: Multiplying $E_{ij}(c)$ to A giving $E_{ij}(c)A$ amounts to performing the row operation $cR_i + R_j \to R_j$ on A..
- Type III: Multiplying E_{ij} to A giving $E_{ij}A$ amounts to performing the row operation $R_i \leftrightarrow R_j$ on A.

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Two matrices A and B are said to be row equivalent if A can be transformed to B by elementary row operations.

Fact: The matrices A and B are row equivalent $\iff$ $B = E_k \cdots E_2 E_1 A$ for some elementary matrices $E_1, E_2, \cdots, E_k$.

Elementary matrices and echelon form

Forward GE: $m \times n$ matrix $A \longrightarrow$ row echelon form $\mathrm{Ref}(A)$

$$\Downarrow$$

$\mathrm{Ref}(A) = E_p \cdots E_2 E_1 A$ for some elementary matrices $E_1, \ldots, E_p$.

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$\mathrm{Rref}(A) = E_k \cdots E_2 E_1 A$ for some elementary matrices $E_1, \ldots, E_k$.

Fact: Let $[A \mid b] \longrightarrow \mathrm{Ref}([A \mid b]) =: [U \mid d]$. Then the system $Ax = b$ and $Ux = d$ are equivalent.

Invertible matrices

An $n \times n$ matrix A is said to be invertible if there exists a matrix B such that $AB = I_n = BA$. Then B is called an inverse of A.

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- For example, the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible since

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.$$

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- The zero matrix $\mathbf{O}$ is never invertible.

Fact: If A is an invertible matrix then its inverse is unique and is denoted by A^{-1}.

Characterization of invertibility

Theorem: Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. A is invertible.
2. $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.
3. $Ax = 0$ has only the trivial solution.
4. The reduced row echelon form of A is I_n.
5. A is a product of elementary matrices.
6. $\mathrm{rank}(A) = n$

Properties of invertible matrices

Fact: Let A and B be two invertible matrices of the same size.

1. If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.
2. The matrix AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
3. The matrix A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$.
4. For any non-negative integer k, the matrix A^k is invertible, and $(A^k)^{-1} = (A^{-1})^k$.

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Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible, and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$ then A is not invertible.

Gauss-Jordan method for computing inverse

Exercise: Let A and B be $n \times n$ matrices. If $AB = I$ or $BA = I$, then show that A is invertible and $B = A^{-1}$.

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Fact: Let A be an $n \times n$ matrix. If $E_p \cdots E_2 E_1 A = I_n$ then $A^{-1} = E_p \cdots E_2 E_1$, where $E_1, \ldots, E_p$ are elementary matrices.

Moral: Elementary row operations that transform A to I_n transform I_n to A^{-1}.

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Gauss-Jordan method:

$$[A \mid I_n] \longrightarrow [I_n \mid X] \Rightarrow A \text{ is invertible and } A^{-1} = X.$$

Example: Gauss-Jordan method

Let $A := \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$. Then $[A \mid I] \to [I \mid A^{-1}]$ gives

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

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$$\Rightarrow A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

LU Factorization

An $n \times n$ matrix A has an LU factorization if $A = LU$, where U is upper triangular and L is unit lower triangular (diagonals are 1).

Fact: If $A \longrightarrow \mathrm{Ref}(A)$ without row interchange then A has an LU factorization.

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- $E_p \dots E_2 E_1 A = \mathrm{Ref}(A) \Rightarrow A = LU$.
- $L := E_1^{-1} E_2^{-1} \dots E_p^{-1}$ and $U := \mathrm{Ref}(A)$.
- Each E_j is unit lower triangular and Type-II $\Rightarrow L$ is unit lower triangular.

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Solution of $Ax = b$ via LU factorization (if exists):

- Compute $A = LU$.
- Solve $Ly = b$ for y - forward substitution.
- Solve $Ux = y$ for x - back substitution.

LU factorization (cont.)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Set $m_{21} := a_{21}/a_{11}$ and $m_{31} := a_{31}/a_{11}$

when $a_{11} \neq 0$ (pivot) and define

$$E_1 := \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}.$$

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Then

$$E_2 E_1 A = E_2 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}.$$

Set $m_{32} := a_{32}^{(1)}/a_{22}^{(1)}$ if $a_{22}^{(1)} \neq 0$ (pivot) and define

LU factorization (cont.)

$E_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}$. Then we have

$$E_3 E_2 E_1 A = E_3 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} = U.$$

.

LU factorization (cont.)

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$$\text{Hence } A = LU, \text{ where } L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}.$$

Examples: LU factorization

Let $A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Then $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Let $A := \begin{bmatrix} 2 & 4 & -1 \\ -4 & -5 & 3 \\ 2 & -5 & -4 \end{bmatrix}$. Then $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

*** End ***