MA 101 (2013)

An algorithm for computing a basis of the null space of a matrix

R. Alam

1. Input: An $m \times n$ matrix A.

   Output: A full column rank matrix X whose columns span the null space of A.

   Solution:

   1. Compute $R = \mathrm{rref}(A)$.
   2. Suppose that R has p-nonzero rows. So it has p-pivot columns. Interchange columns of R (this means choose a permutation matrix P) so that
   $$RP = \begin{bmatrix} I_p & F \\ 0 & 0 \end{bmatrix} = \text{column interchanged form of } R,$$
   where I_p is the identity matrix of size p.
   3. Set $Y := \begin{bmatrix} -F \\ I_{n-p} \end{bmatrix}$, where I_{n-p} is the identity matrix of size $n-p$.
   4. Now interchange rows of Y according to the permutation P. This means compute
   $$X := PY.$$
   Then $\mathrm{rank}(X) = n - p$ and $RX = RPY = 0$. Thus columns of X span the null space of R and hence the null space of A.

2. **Example:** Find the basis of the null space and the range space of the matrix
$$A := \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

   Solution: We have
   $$R = \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
   Interchanging 2nd and 3rd columns of R, we have
   $$RP = \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & F \\ 0 & 0 \end{bmatrix}.$$

Now define

$$Y := \begin{bmatrix} -F \\ I_{n-p} \end{bmatrix} = \left[\begin{array}{cc} -3 & 1 \\ 0 & -1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right],$$

where $p = 2$ and $n = 4$.

Finally, interchange 2nd and 3rd row of Y to obtain X, that is,

$$X = PY = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$

which gives a basis of the null space of A.

Since 1st and 3rd columns of R are the pivot columns, the 1st and 3rd columns of A form a basis of the column space (range space) of A.