MA 201 Complex Analysis
Lecture 15: Residues Theorem

Residues

Question: Let γ is a simple closed contour in a simply connected domain D and let z_0 doesn't lie on γ. If f has singularity only at z_0 then what could be the value for $\int_\gamma f(z)dz$?

- **Recall: Laurent's Theorem:** Suppose that $0 \leq r < R$. Let f be analytic in the annulus $A = \text{ann}(z_0, r, R)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the convergence is absolute and uniform in $\overline{\text{ann}(z_0, r_1, R_1)}$ if $r < r_1 < R_1 < R$. The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

for any $r < s < R$.

Moreover, this series is unique.

- Put $n = -1$. The **answer of the above question** is $2\pi i a_{-1}$.

Residues

Definition: Let f have an isolated singularity at $z = z_0$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be the Laurent series expansion about z_0 then the residue at f is the coefficient a_{-1}.

- We denote $\text{Res}(f, z_0) = a_{-1}$.
- If f has a removable singularity at $z = z_0$ then $\text{Res}(f, z_0) = 0$.
- If $f(z) = \dfrac{\sin z}{z}$ then $\text{Res}(f, 0) = 0$.
- Let $f(z) = e^{\frac{2}{z}}$ and $g(z) = e^{\frac{1}{z^2}}$. Then $\text{Res}(f, 0) = 2$ and $\text{Res}(g, 0) = 0$.

Residue at poles

- If f has a simple pole (pole of order one) at $z = z_0$ then

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

in this case $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$.

- If f has a pole of order m at $z = z_0$ then $f(z) = \frac{g(z)}{(z-z_0)^m}$, $g(z_0) \neq 0$. Since g is analytic at z_0 we can write

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \cdots .$$

So

$$f(z) = \frac{g(z)}{(z - z_0)^m} = \frac{b_0}{(z - z_0)^m} + \frac{b_1}{(z - z_0)^{m-1}} + \cdots \frac{b_{m-1}}{(z - z_0)} + \sum_{k=0}^{\infty} b_{m+k} (z - z_0)^k.$$

Now $\text{Res}(f, z_0) = b_{m-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z)$.

Residue at poles

- If f has a pole of order m at $z = z_0$ then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [f(z)(z - z_0)^m].$$

- Let $f(z) = \dfrac{z}{(z-1)(z+1)^2}$ then

$$\mathrm{Res}(f, 1) = \lim_{z \to 1} f(z)(z-1) = \frac{1}{4}$$

and

$$\mathrm{Res}(f, -1) = \lim_{z \to -1} \frac{d}{dz} [f(z)(z+1)^2] = -\frac{1}{4}.$$

Cauchy residue theorem

Cauchy residue theorem: Let f be analytic inside and on a simple closed contour γ (positive orientation) except for finite number of isolated singularities $a_1, a_2 \cdots a_n$. If the points $a_1, a_2 \cdots a_n$ does not lie on γ then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, a_k).$$

Proof. Apply **Cauchy's theorem for multiply connected domain**.

- $\int_\gamma f(z)dz = (2\pi i)\times$ sum of the residues of f at all singular points that are enclosed in γ.
- $\displaystyle\int_{|z|=1} \frac{1}{z(z-2)} dz = 2\pi i \times \text{Res}(f, 0)$. (The point $z = 2$ does not lie inside unit circle.)

Argument Principele

Definition: A function f is said to be **meromorphic** in a domain D if f is analytic throughout D except for poles.

- Suppose f is meromorphic inside a closed contour γ, analytic on γ has no zero on γ. Let Γ be the image of γ under f i.e. $\Gamma = f(\gamma)$ then Γ is a closed contour (not necessarily simple).
- As z traverses γ in positive direction, its image $w = f(z)$ traverses Γ in a particular direction that determines the orientation of Γ.
- Fix $f(z_0) = w_0 \in \Gamma$. Let $\phi_0 = \arg w_0$. Take $w \in \Gamma$. Vary $\arg w$ continuously starting with the value ϕ_0.
- When w returns to the point w_0 (in this case z traverses from z_0 to z_0), $\arg w$ assumes a particular value of $\arg w_0$ (say ϕ_1).
- The change in $\arg w$ (independent of the point w_0) is $\phi_1 - \phi_0$ which is an integral multiple of 2π.
- The integer $\frac{1}{2\pi}(\phi_1 - \phi_0)$ represents orientation and the number of times the point w winds around the origin called the **winding number**.

Argument Principle

Question: Can we determine the winding number using the number of zeros and poles of f lying interior to a closed contour γ?

The Answer is given by Argument principle.

Argument principle: Suppose a function $f(z)$ is meromorphic in the domain interior to a positively oriented simple closed contour γ such that

- $f(z)$ is analytic and nonzero on γ
- $Z =$ No. of **zeros** of f counted according to multiplicity and
  $P =$ No. of **poles** of f counted according to multiplicity.

Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = (Z - P).$$

Argument Principle

The contour integral $\displaystyle\int_\gamma \frac{f'(z)}{f(z)}\,dz$ can be interpreted in following (**informal**) ways:

- as the **total change in the argument** of $f(z)$ as z travels around γ, explaining the name of the theorem; since

$$\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)},$$

  then the integration of $\frac{f'(z)}{f(z)}$ over γ gives

$$\log f|_\gamma = [\cancelto{0}{\log|f(z)|} + i\arg f(z)]|_\gamma.$$

- as $2\pi i$ times the **winding number** of the path $f(\gamma)$ around the origin, using the substitution $w = f(z)$ one has

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_{f(\gamma)} \frac{1}{w}\,dw.$$

Argument Principle

Proof. Suppose that f is analytic and has a zero of order m at $z = a$. So $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$. So $\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$. So by residue theorem,

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i m.$$

Suppose that f has a pole of order n at $z = a$. So $f(z) = (z-a)^{-n} g(z)$ where $g(a) \neq 0$. So $\frac{f'(z)}{f(z)} = \frac{-n}{z-a} + \frac{g'(z)}{g(z)}$. So by residue theorem,

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i(-n).$$

Combining the above two results we have

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 2\pi i(Z - P).$$

- Evaluate: $\displaystyle\int_{|z-\frac{\pi}{2}|=1} \tan z\, dz$.
- Evaluate: $\displaystyle\int_{|z|=1} \frac{dz}{\sin z}$. [Hint. $f(z) = \tan(z/2)$]

Rouché's Theorem

Theorem: Suppose f, g be two analytic functions inside and on a simple closed contour γ such that $|f(z)| > |g(z)|$ at each point on γ. Then $f(z)$ and $f(z) + g(z)$ have same number of zeros, counted according to their multiplicity inside γ.

Example: Determine the number of zeros of the equation $z^7 - 4z^3 + z - 1 = 0$ inside the circle $|z| = 1$.

Take $f(z) = -4z^3$; $g(z) = z^7 + z - 1$. Then $|f(z)| = 4$ and $|g(z)| \leq 3$ when $|z| = 1$. Since f has three zeros inside $|z| = 1$, by Rouché$'s$ theorem, the equation $z^7 - 4z^3 + z - 1 = 0$ has three zeros inside the circle $|z| = 1$.