MA 201 Complex Analysis
Lecture 7: Complex Integration

Integral of a complex valued function of real variable:

- **Definition:** Let $f : [a, b] \to \mathbb{C}$ be a function. Then $f(t) = u(t) + iv(t)$ where $u, v : [a, b] \to \mathbb{R}$.Define,

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

If $U' = u$ and $V' = v$ and $F(t) = U(t) + iV(t)$ then by fundamental theorem of calculus $\int_a^b f(t)dt = F(b) - F(a)$.

- For $\alpha \in \mathbb{R}$, $\int_a^b e^{i\alpha t}dt = \dfrac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}$.
- $\int_0^1 (1 + it)^2\, dt = \int_0^1 (1 - t^2)\, dt + i\int_0^1 2t\, dt = \dfrac{2}{3} + i.$
- If $f : [a, b] \to \mathbb{C}$ piecewise continuous then $\int_a^b f(t)dt$ exists.

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- $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

Proof: Let $\int_a^b f(t)dt = Re^{i\theta}$ then,

$$\begin{aligned} R &= e^{-i\theta}\int_a^b f(t)dt = \int_a^b e^{-i\theta}f(t)dt \\ &= \text{Re}\left(\int_a^b e^{-i\theta}f(t)dt\right) = \int_a^b \text{Re}\,(e^{-i\theta}f(t))dt. \end{aligned}$$

Therefore,

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- A curve γ is called a **smooth curve** if γ is differentiable and γ' is continuous and nonzero for all t.

- A **contour/piecewise smooth curve** is a curve that is obtained by joining finitely many smooth curves end to end.

- $\gamma_1(t) = e^{it}, t \in [0, 1]; \quad \gamma_2(t) = (1 - t)a + tb, \; t \in [0, 1].$

- **Definition:** A curve γ is **simple** if it does not intersect itself except possibly at end points. That means $\gamma(t_1) \neq \gamma(t_2)$ when $a < t_1 < t_2 < b$.

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- **Orientation:** Let γ be a simple closed contour with parametrization $\gamma(t),\ t \in [a, b]$. As t moves from a to b, the curve γ moves in a specific direction called the orientation of the curve induced by the parametrization.

- **Convention:** If the interior bounded domain of γ is kept on the left as t moves from a to b, then we say the orientation is in the **positive sense** (counter clockwise or anticlockwise sense). Otherwise γ is oriented **negatively** (clockwise direction).

- Let $\gamma : [a, b] \to \mathbb{C}$ be a curve then the curve with the reverse orientation is denoted as $-\gamma$ and is defined as

$$-\gamma : [a, b] \to \mathbb{C}, ; \quad -\gamma(t) = \gamma(b + a - t).$$

- $\gamma(t) = e^{it},\ t \in [0, 2\pi]$ (Positive orientation)

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- Let γ be a piecewise smooth curve defined on $[a, b]$. The length of γ is given by
$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

- **Definition:** Let $\gamma(t)$; $t \in [a, b]$, be a contour and f be complex valued continuous function defined on a set containing γ then the **line integral or the contour integral** of f along the curve γ is defined by
$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

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- Let γ be a piecewise smooth curve defined on $[a, b]$. The length of γ is given by
$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

- **Definition:** Let $\gamma(t)$; $t \in [a, b]$, be a contour and f be complex valued continuous function defined on a set containing γ then the **line integral or the contour integral** of f along the curve γ is defined by
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Example: Let $f(z) = \bar{z}$.

- If $\gamma_1(t) = e^{it}$, $t \in [0, \pi]$ then,

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- (The fundamental integral) For $a \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{Z}$

$$\int_{C_{a,r}} (z-a)^n dz = \begin{cases} 0 & \text{if} \quad n \neq -1 \\ 2\pi i & \text{if} \quad n = -1 \end{cases}$$

  where $C_{a,r}$ denotes the circle of radius r centered at a.

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$$\int_\gamma [\alpha f \pm g](z)dz = \alpha \int_\gamma f(z)dz \pm \int_\gamma g(z)dz.$$

- Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and $a < c < b$. If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ then

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ML-inequality:

- Let f be a piecewise continuous function and let γ be a contour. If $|f(z)| \leq M$ for all $z \in \gamma$ and L =length of γ then

$$\left|\int_\gamma f(z)dz\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq M \int_a^b |\gamma'(t)|dt = ML.$$

- Let $\gamma(t) = 2e^{it}, t \in [0, \frac{\pi}{2}]$ and $f(z) = \dfrac{z+4}{z^3-1}$. Then by ML-ineuqality

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Antiderivatives

- Answer to the **Question:** When a line integral of f does not depend on path?

- **Definition:** The **antiderivative or primitive** of a continuous function f in a domain D is a function F such that $F'(z) = f(z)$ for all $z \in D$. The primitive of a function is **unique** up to an additive constant.

- **Theorem:** Let f be a continuous function defined on a domain D and $f(z)$ has antiderivative $F(z)$ in D. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2 the value of the integral

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- **Proof.** Suppose that C is given by a map $\gamma : [a, b] \to \mathbb{C}$. Then $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t)$. Hence

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \frac{d}{dt}F(\gamma(t))dt \\ &= F(\gamma(a)) - F(\gamma(b)) = F(z_2) - F(z_1).\end{aligned}$$

- When such F exists we write

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} F'(z)dz = F(z_1) - F(z_2).$$

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$$\int_\gamma \frac{dz}{z^n} = -(n-1)\left(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}}\right).$$

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