MA 201 Complex Analysis
Lecture 6: Elementary functions

The Exponential Function

Recall:

- Euler's Formula: For $y \in \mathbb{R}$, $e^{iy} = \cos y + i \sin y$
- and for any $x, y \in \mathbb{R}$, $e^{x+y} = e^x e^y$.

Definition: If $z = x + iy$, then e^z or $\exp(z)$ is defined by the formula

$$e^z = e^{(x+iy)} = e^x(\cos y + i \sin y).$$

Image of following sets under exponential function:

- $\{(x, y_0) : x \in \mathbb{R}\} \longmapsto \{(r, \theta) : r = e^x, x \in \mathbb{R}, \theta = y_0\}$.
- $\{(x_0, y) : y \in \mathbb{R}\} \longmapsto \{(r, \theta) : r = e^{x_0}, \theta \in \mathbb{R}\}$.
- $\{(x, y) : a \leq x \leq b, c \leq y \leq d\} \longmapsto \{(r, \theta) : e^a \leq r \leq e^b, c \leq \theta \leq d\}$.

Properties of Exponential Function

- $e^0 = 1$
- $\overline{e^z} = e^{\bar{z}}$, $|e^z| \leq e^{|z|}$.
- $e^z \neq 0$, for all $z \in \mathbb{C}$. Look at $|e^z| = |e^x||e^{iy}| = e^x \neq 0$.
- For all $z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$.
- Indeed, if $z = x + iy$, $w = s + it$ then,

$$\begin{aligned} e^{z+w} &= e^{(x+s)+i(y+t)} = e^{(x+s)}[\cos(y+t) + i\sin(y+t)] \\ &= e^x e^s[(\cos y \cos t - \sin y \sin t) + i(\sin y \cos t + \cos y \sin t)] \\ &= [e^x(\cos y + i\sin y)][e^s(\cos t + i\sin t)] \\ &= e^z e^w. \end{aligned}$$

Properties of Exponential function

- A function $f : \mathbb{C} \to \mathbb{C}$ is called **periodic** if there is a $w \in \mathbb{C}$ (called a **period**) such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$.

- e^z is periodic function with period $2\pi i$.

- e^z is not injective *unlike* real exponential.

- Since $e^z = e^x \cos y + ie^x \sin y$ satisfies C-R equation on $\mathbb{C}$ and has continuous first order partial derivatives. Therefore e^z is an entire function.

- In fact $\frac{\mathrm{d}}{\mathrm{d}\, z} e^z = \frac{\partial}{\partial x}(e^x \cos y) + i \frac{\partial}{\partial x}(e^x \sin y) = e^z$.

Trigonometric Functions

Define

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Properties:

- $\sin^2 z + \cos^2 z = 1$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$
- $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$
- $\sin(z + 2k\pi) = \sin z$, and $\cos(z + 2k\pi) = \cos z$
- $\sin z = 0 \Longleftrightarrow z = n\pi$ and $\cos z = 0 \Longleftrightarrow z = (n + \frac{1}{2})\pi, \ , \ n \in \mathbb{Z}$.

Trigonometric functions

- $\sin z, \cos z$ are **entire** functions.
- $\frac{d}{dz}(\sin z) = \cos z$ and $\frac{d}{dz}(\cos z) = -\sin z$.
- $\sin z$ and $\cos z$ are unbounded functions.

Define:

- $\tan z = \dfrac{\sin z}{\cos z}$
- $\cot z = \dfrac{\cos z}{\sin z}$
- $\sec z = \dfrac{1}{\cos z}$
- $\csc z = \dfrac{1}{\sin z}$.

Trigonometric functions

- **Hyperbolic Trigonometric functions:** Define

$$\sinh z = \frac{e^z - e^{-z}}{2}; \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

- **Properties:**
  - $\sinh z, \cosh z$ are **entire** functions.
  - $\cosh^2 z - \sinh^2 z = 1.$
  - $\sinh(-z) = -\sinh z,\ \cosh(-z) = \cosh z,$
  - $\sinh(z + 2k\pi) = \sinh z,\ \cosh(z + 2k\pi) = \cosh z,\ \ k \in \mathbb{Z}.$
  - $\sinh(iz) = i \sin z$ and $\cos(iz) = \cos z$
  - $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$
  - $\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$

Complex Logarithm

- Note that e^z is an **onto** function from $\mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In fact if $w \in \mathbb{C}^*$ then $w = |w|e^{i\theta}$ where $\theta \in (-\pi, \pi]$. If we set $z = \log|w| + i\theta$ then,
$$e^z = e^{\log|w| + i\theta} = e^{\log|w|}e^{i\theta} = w.$$

- e^z is not an **injective** function as $e^{z+2\pi ik} = e^z, \; k \in \mathbb{Z}$. But *if we restrict the domain then it becomes injective.*

- In particular if
$$H = \{z = x + iy : -\pi < y \leq \pi\}$$
then $z \to e^z$ is a **bijective** function from H to $\mathbb{C} \setminus \{0\}$.

Question: What is the inverse of this function?

Complex Logarithm

Definition: For $z \in \mathbb{C}^*$, **define** $\log z = \ln|z| + i \arg z$.

- $\ln|z|$ stands for the real logarithm of $|z|$.
- Since $\arg z = \operatorname{Arg} z + 2k\pi$, $k \in \mathbb{Z}$ it follows that $\log z$ is not well defined as a function. (**multivalued**)
- For $z \in \mathbb{C}^*$, the **principal value** of the logarithm is defined as $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$.
- $\operatorname{Log} : \mathbb{C}^* \to \{z = x + iy : -\pi < y \leq \pi\}$ is well defined (**single valued**).
- $\operatorname{Log} z + 2k\pi i = \log z$ for some $k \in \mathbb{Z}$.

Complex Logarithm

- If $z \neq 0$ then $e^{\text{Log } z} = e^{\ln|z| + i \text{ Arg} z} = z$
- What about Log (e^z)?.
- Suppose x is a positive real number then Log $x = \ln x + i$ Arg$x = \ln x$.
- Log $i = \ln|i| + i\frac{\pi}{2} = \frac{i\pi}{2}$,
- Log $(-1) = \ln|-1| + i\pi = i\pi$,
- Log $(-i) = \ln|-i| + i\frac{-\pi}{2} = -\frac{i\pi}{2}$,
- Log $(-e) = 1 + i\pi$

Complex Logarithm

- The function Log z is **not continuous** on the **negative real axis** $\mathbb{R}^- = \{z = x + iy : x < 0, y = 0\}$.

  To see this consider the point $z = -\alpha$, $\alpha > 0$. Consider the sequences

  $$\{a_n = \alpha e^{i(\pi - \frac{1}{n})}\} \text{ and } \{b_n = \alpha e^{i(-\pi + \frac{1}{n})}\}.$$

  Then

  $$\lim_{n\to\infty} a_n = z = \lim_{n\to\infty} b_n$$

  but

  $$\lim_{n\to\infty} \text{Log } a_n = \lim_{n\to\infty} \ln\alpha + i(\pi - \frac{1}{n}) = \ln\alpha + i\pi$$

  and

  $$\lim_{n\to\infty} \text{Log } b_n = \ln\alpha - i\pi.$$

Complex Logarithm

- The function $z \to \text{Log } z$ is **analytic** on the set $\mathbb{C}^* \setminus \mathbb{R}^-$. Let $z = re^{i\theta} \neq 0$ and $\theta \in (-\pi, \pi)$. Then

$$\text{Log } z = \ln r + i\theta = u(r,\theta) + iv(r,\theta)$$

with $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$. Then

$$u_r = \frac{1}{r} v_\theta = \frac{1}{r} \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta .$$

- The identity

$$\text{Log } (z_1 z_2) = \text{Log} z_1 + \text{Log } z_2$$

is not always valid. However, the above identity is true if and only if $\text{Arg } z_1 + \text{Arg } z_2 \in (-\pi, \pi]$ (why?).

Complex Logarithm

- **Branch of a multiple valued function:** Let F be a multiple valued function defined on a domain D. A function f is said to be a **branch** of the multiple valued function F if in a domain$D_0 \subset D$ if $f(z)$ **is single valued and analytic in** D_0.
- **Branch Cut:** The portion of a line or a curve introduced in order to define a branch of a multiple valued function is called **branch cut**.
- **Branch Point:** Any point that is common to all branch cuts is called a **branch point**.

Complex Exponents

Let $w \in \mathbb{C}$. For any $z \neq 0$, define

$$z^w = exp(w \log z),$$

where "exp" is the exponential function and log is the multiple valued logarithmic function.

- z^w is a multiple valued function.
- $i^i = \exp[i \log i] = \exp[i(\log 1 + i\frac{\pi}{2})] = \exp(-\frac{\pi}{2})$.