MA 201 Complex Analysis
Lecture 5: Analytic functions

Analytic functions

- **Definition:** A function f is called **analytic** at a point $z_0 \in \mathbb{C}$ if there exist $r > 0$ such that f is differentiable at every point $z \in B(z_0, r)$.
- A function is called analytic in an open set $U \subseteq \mathbb{C}$ if it is analytic at each point U.
- An **entire** is a function which is analytic on the whole complex plane $\mathbb{C}$.
- For $n \in \mathbb{N}$ and complex numbers $a_0, \ldots, a_n$ the polynomial $f(z) = \sum_{k=0}^{n} a_k z^k$ entire.
- The function $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$ (hence not entire).
- Analyticity $\implies$ Differentiability, where as Differentiability $\not\Longrightarrow$ Analyticity.
- **Example:** The function $f(z) = |z|^2$ is differentiable only at $z = 0$ however it is not analytic at any point.

Analytic functions

Let $f(z) = u(x, y) + iv(x, y)$ be defined on an open set $D \subseteq \mathbb{C}$.

- f is analytic on $D \implies f$ satisfies CR Equation on D.
- f satisfies CR Equation on D and u, v has continuous first order partial derivatives on $D \implies f$ is differentiable on $D \implies f$ is analytic on D
- Suppose f, g are analytic in an open set D. Then $f \pm g, fg, \dfrac{f}{g}\ (g \neq 0), \alpha f\ (\alpha \in \mathbb{C})$ are analytic on D.
- Composition of analytic functions is analytic.
- Let f is analytic in a domain D. If the **real part** or **imaginary part** or **argument** or **modulus** of f is constant then f is constant in D.

Harmonic Functions

- **Harmonic functions:** A real valued function $\phi(x,\ y)$ is said to be **harmonic** in a domain D if
  1. all the partial derivatives up to second order exists and continuous on D
  2. ϕ satisfies the Laplace equation $\phi_{xx}(x,\ y) + \phi_{yy}(x,\ y) = 0$ at each point of D.

- **Theorem:** If $f(z) = u(x,\ y) + i\ v(x,\ y)$ is analytic in a domain D, then the functions $u(x,\ y)$ and $v(x,\ y)$ are harmonic in D.
  Proof: Since f is analytic in D, f satisfies the CR equations $u_x = v_y$ and $u_y = -v_x$ in D.
  Now, it gives that $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Consequently, $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$. Therefore, u is harmonic in D. Similarly, one can show that v is harmonic in D.

Note: We have used the fact that all the second order partial derivatives $(u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy})$ exists which will follow from the fact that "if f is analytic at a point then its derivatives of all orders exists at that point".(Prove Later!)

Harmonic Conjugate

Let D be a domain and $u : D \to \mathbb{R}$ is harmonic. Does there exists a harmonic function $v : D \to \mathbb{R}$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic in D? If such harmonic function $v : D \to \mathbb{R}$ exists then v is called the **harmonic conjugate** of u.

- The function $v(x, y) = 2xy$ is a harmonic conjugate of $u(x, y) = x^2 - y^2$ in $\mathbb{C}$. The function $f(z) = z^2 = (x^2 - y^2) + i\,(2xy)$ is analytic in $\mathbb{C}$.
- Does harmonic conjugate v always exist for a given harmonic function u in a domain D? Answer: 'No'.
- The function $u(x, y) = \log(x^2 + y^2)^{\frac{1}{2}}$ is harmonic on $G = \mathbb{C} \setminus \{0\}$ and it has no harmonic conjugate on G.
- **Question:** Under what condition harmonic conjugate v exists for a given harmonic function u in a domain D?
- **Theorem:** Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate in G.

Harmonic Conjugate

- **Construction of a harmonic conjugate** Let $u(x,y) = x^2 - y^2$. We have to find the harmonic conjugate of u.
- **Step 1:** Check that u is harmonic: clearly $u_{xx} + u_{yy} = 2 - 2 = 0$.
- **Step 2:** Calculate u_x and u_y: $u_x(x,y) = 2x$ and $u_y(x,y) = -2y$. Since the conjugate harmonic function v satisfied CR equations we have

$$u_x(x,y) = v_y(x,y) = 2x \implies v(x,y) = \int u_x(x,y)\,dy + \phi(x) = 2xy + \phi(x).$$

- Consider

$$v_x(x,y) = 2y + \phi'(x) = -u_y(x,y) = 2y \implies \phi'(x) = 0.$$

So $v(x,y) = 2xy + c$, where c is a constant.

So $f(x,y) = u(x,y) + iv(x,y) = x^2 - y^2 + 2ixy + ic = z^2 + ic$ is analytic.

Harmonic conjugate

- Given a harmonic function u. Suppose the harmonic conjugate of u exists. Is it unique?
  Ans: Yes, it is unique up to an additive constant.
- **Proof.** Let v_1 and v_2 be two harmonic conjugates of u. Then $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are analytic. Then $f_1 - f_2 = i(v_1 - v_2)$ is analytic. So $v_1 = C + v_2$.
- A function $f(z) = u(x, y) + iv(x, y)$ is analytic if and only if v is the harmonic conjugate of u.