Some Basic Definitions

Interior Points and Open set

- **Open ball/disc**: Let $z_0 \in \mathbb{C}$ and $r > 0$ then, $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is called an open disc centered at z_0 with radius r.
- **Interior point:** A point z_0 is called an **interior point** of a set $S \subset \mathbb{C}$ if we can find an $r > 0$ such that $B(z_0, r) \subset S$. We denote set of all interior points of S by S^o.
- **Open Set:** A set $S \subset \mathbb{C}$ is **open** if every $z_0 \in S$ there exists $r > 0$ such that $B(z_0, r) \subset S$.
- **Exercise:** Show that a set S is an open set if and only if every point of S is an interior point.
- **Deleted Neighborhood of z_0 :** Let $z_0 \in \mathbb{C}$ and $r > 0$ then, $B(z_0, r) - \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ is called the **deleted neighborhood** of z_0.

Open set (Some Remarks)

- Every open ball is an open set.
- **Explanation:**
  - Let $B(z_0, r_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\}$ be an open ball centered at z_0 with radius r_0.
  - To prove $B(z_0, r_0)$ is an open set, we need to show that any $z \in B(z_0, r_0) \ \exists, ; \ r > 0$ such that $B(z, r) \subset B(z_0, r_0)$. That can be done easily.
  - If we take $r = \min\{|z_0 - z|, r_0 - |z_0 - z|\}$ then for any $w \in B(z, r)$ we have

$$|z_0 - w| \leq |z_0 - z| + |z - w| < r_0.$$

    This show that $w \in B(z_0, r_0)$ and hence $B(z, r) \subset B(z_0, r_0)$.
- Similarly we can show that any deleted neighborhood $B(z_0, r) - \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ is also an open set.
- From above observation it is obvious that every point in the ball $B(z_0, r_0)$ is an interior point of $B(z_0, r_0)$..

Open set (Some Remarks continue....)

- Let $A_1, A_2, \ldots, A_k, \ldots$ be a countable (infinite) collection of open subsets of $\mathbb{C}$.
- If $\bigcup_{k=1}^{\infty} A_k = A$ then A is an open set. Proof is easy.
- If $\bigcap_{k=1}^{\infty} A_k = B$ then B may not be an open set. Why?
- **Example:** If we take $A_k = B(z, 1/k)$ for $k = 1, 2, 3, \ldots$ then $B = \{z\}$. Since a singleton set can not be an open set, therefore B is not an open set
- However if $\bigcap_{k=1}^{n} A_k = A_0$ then A_0 is an open set. How?
- **Explanation:** If $z \in A_0$ then $z \in A_k$ for all $k = 1, \ldots n$. Since all A_k are open therefore $\exists\ r_k$ such that $B(z, r_k) \subset A_k$ for all $k = 1, \ldots n$.
- So if we choose $r = \min\{r_1, \ldots r_n\}$ then $B(z, r) \subset B(z, r_k) \subset A_k$ for all $k = 1, \ldots n$. This implies that $B(z, r) \subset A_0$.

Connected Set and Domain

- Let $[z, w] = \{(1-t)z + tw : 0 \leq t \leq 1\}$ be the line segment in the complex plane joining the points z and w.
- **Connected Set:** A set $S \subset \mathbb{C}$ is said to be **connected** if each pair of points z_1 and z_2 in S can be joined by a polygonal line consisting of a finite number of line segments joined end to end that lies entirely in S.
- In other words a set S is connected if for each pair of points z_1 and z_2 in S there exists finitely many points $w_1, \ldots, w_k$ such that $[z_1, w_1] \cup [w_1, w_2] \cup \ldots \cup [w_k, z_2] \subset S$.
- **Domain/Region:** An open, connected set is called a **domain**.

Closed Disc, Bounded Set and Boundary Points

- **Closed disc**: Let $z_0 \in \mathbb{C}$ and $r > 0$ then, $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ is called a closed disc centered at z_0 with radius r.
- **Bounded Set:** A set $S \subset \mathbb{C}$ is **bounded** if there exists a $K > 0$ such that $|z| < K \ \forall \ z \in S$. We say S is **unbounded** if S is not bounded.
- **Boundary points:** If $B(z_0, r)$ contains points of S and points of S^c every $r > 0$, then z_0 is called a **boundary point** of a set S. We denote set of all boundary points of S by ∂S.
- Some examples:
  - Set of all boundary points of open disc $B(z, r)$ or closed disc $D(z, r)$ is the circle $\{w : |z - w| = r\}$.
  - If $S = \{w : |z - w| = r\}$ then $\partial S = S$.
  - If S is finite set then $\partial S = S$.
  - If $S = \{1 + i\frac{1}{n} : n \in \mathbb{N}\}$ then $\partial S = S \cup \{1\}$
- **Exterior points:** If a point is not an interior point or boundary point of S, it is an exterior point of S.

Limit Points, Closed Set and Compact Set

- **Limit point/Accumulation point:** Let ζ is called an **limit point** of a set $S \subset \mathbb{C}$ if every deleted neighborhood of ζ contains at least one point of S. We denote set of all limit points of S by S'.
- Some examples:
    - Set of all limit points of open disc $B(z, r)$ or closed disc $D(z, r)$ is $D(z, r)$.
    - If S is finite set then $S' = \emptyset$. (why?)
    - If $S = \{w : |z - w| = r\}$ then $S' = S$.
    - If $S = \{i\frac{1}{n} : n \in \mathbb{N}\}$ then $S' = \{0\}$. (why?)
- **Closed Set:** A set $S \subset \mathbb{C}$ is **closed** if S contains all its limit points, i.e. $S' \subset S$.
- **Exercise:** Show that a set S is closed if and only if S^c is open.

Closure of a Set and Compact Set

- **Closure of a Set:** The **closure** of a set $S \subset \mathbb{C}$, denoted by $\bar{S}$, defined by the set S together with all its limit points. In other words $\bar{S} = S' \cup S$.
- Some examples:
  - If S is finite set then $\bar{S} = S$.
  - Closure of open disc $B(z, r)$ or closed disc $D(z, r)$ is $D(z, r)$. If $S = \{w : |z - w| = r\}$ then $S' = S$.
  - If $S = \{i\frac{1}{n} : n \in \mathbb{N}\}$ then $\bar{S} = S \cup \{0\}$
- **Exercise:** Show that a set S is closed if and only if $\bar{S} = S$.
- **Compact Set:** A set $K \subset \mathbb{C}$ is **compact** if K is a closed and bounded subset of $\mathbb{C}$.
- All closed discs, finite sets, circles are example of compact sets.