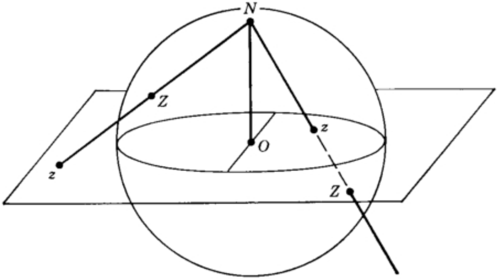
MA 201 Complex Analysis
Lecture 18 and 19: Conformal Mapping

The extended complex plane and Riemann Sphere

- Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere $\mathbb{R}^3$ and let $N = (0, 0, 1)$ denote the "north pole" of S^2.
- Identify $\mathbb{C}$ with $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ so that $\mathbb{C}$ cuts S^2 in the equator.
- For each $z \in \mathbb{C}$ consider the straight line in $\mathbb{R}^3$ through z and N. This straight line intersects the sphere in exactly one point $Z \neq N$.



The extended plane and its spherical representation

- **Question:** What happens to Z as $|z| \to \infty$?
- **Answer:** The point Z approaches N.
- Hence we identify N with the point ∞ in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Thus the extended complex plane $\mathbb{C}_\infty$ is represented as the sphere S^2 is known as **Riemann Sphere**.
- Let $z = x + iy \in \mathbb{C}$ and let $Z = (X_1, X_2, X_3)$ be the corresponding point on S^2.
- We want to find X_1, X_2 and X_3 in terms of x and y.

The extended plane and its spherical representation

- The parametric equation of straight line in $\mathbb{R}^3$ through z and N is given by

$$\{tN + (1-t)z : t \in \mathbb{R}\} \text{ i.e. } \{((1-t)x, (1-t)y, t) : t \in \mathbb{R}\}.$$

- Since this line intersects S^2 at $Z = ((1-t_0)x, (1-t_0)y, t_0)$, we have

$$1 = (1-t_0)^2 x^2 + (1-t_0)^2 y^2 + t_0^2 \Longrightarrow t_0 = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

- Thus,

$$X_1 = \frac{2x}{x^2+y^2+1},\ X_2 = \frac{2y}{x^2+y^2+1} \text{ and } X_3 = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

The extended plane and its spherical representation

- Now we will write x and y in terms of X_1, X_2 and X_3.
- From the last observation that

$$1 - X_3 = \frac{2}{x^2 + y^2 + 1}.$$

- Given the point $Z \in S^2$ the corresponding point $z = x + iy \in \mathbb{C}$ is

$$x = \frac{X_1}{1 - X_3} \quad \text{and} \quad y = \frac{X_2}{1 - X_3}.$$

- The correspondence between S^2 and $\mathbb{C}_\infty$ defined by

$$Z = (X_1, X_2, X_3) \mapsto z = (x + iy)$$

is called **stereographic projection**.

Conformal Mapping

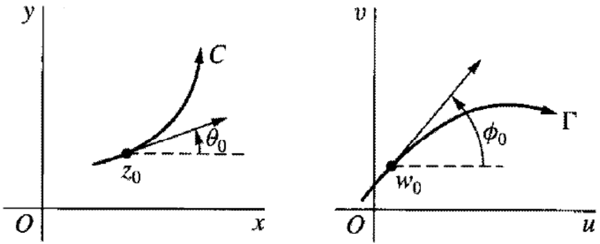
Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve in a domain D. Let $f(z)$ be a function defined at all points z on γ. Let C denotes the image of γ under the transformation $w = f(z)$. The parametric equation of C is given by $C(t) = w(t) = f(\gamma(t)),\ t \in [a, b]$.

- Suppose that γ passes through $z_0 = \gamma(t_0)$, $(a < t_0 < b)$ at which f is analytic and $f'(z_0) \neq 0$. Then

$$w'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0).$$

- That means

$$\arg\ w'(t_0) = \arg\ f'(z_0) + \arg\ \gamma'(t_0).$$

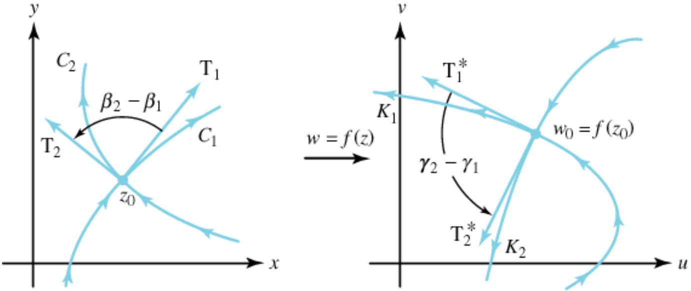


Let $C_1, C_2 : [a, b] \to \mathbb{C}$ be two smooth curves in a domain D passing through z_0. Then by above we have

$$\arg\ w_1'(t_0) = \arg\ f'(z_0) + \arg\ C_1'(t_0) \text{ and } \arg\ w_2'(t_0) = \arg\ f'(z_0) + \arg\ C_2'(t_0)$$

that means

$$\arg\ w_2'(t_0) - \arg\ w_1'(t_0) = \arg\ C_2'(t_0) - \arg\ C_1'(t_0).$$



Conformal Mapping

This summarizes as follows

- **Theorem:**If $f : D \to \mathbb{C}$ is analytic then f preserves angles at each point z_0 of D where $f'(z) \neq 0$.
- **Definition:** A map $w = f(z)$ is said to be conformal if it preserves angel between oriented curves in magnitude as well as in orientation.
- if f is analytic and $f'(z) \neq 0$ for any z then f is conformal. The converse of this statement is also true.
- Let $f(z) = e^z$. Then f is a conformal at every point in $\mathbb{C}$ as $f'(z) = f(z) = e^z \neq 0$ for each $z \in \mathbb{C}$.
- Let $f(z) = \sin z$. Then f is a conformal map on $\mathbb{C} \setminus \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$.
- Let $f(z) = \bar{z}$. Then f is not a conformal map as it preserves only the magnitude of the angle between the two smooth curves but not **orientation**. Such transformations are called **isogonal mapping**.

Scale Factors

- **Definition:** If f is analytic at z_0 and $f'(z_0) = 0$ then the point z_0 is called a critical point of f.
- The behavior of an analytic function in a neighborhood of critical point is given by the following theorem:
- **Theorem:** Let f be analytic at z_0. If $f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$ and $f^k(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies the angle at the vertex z_0 by a factor k.
- **Proof.** Since f is analytic at z_0 we have

$$
\begin{aligned}
f(z) &= f(z_0) + (z - z_0)f'(z_0) + \frac{1}{2!}(z - z_0)^2 f''(z_0) + \cdots \\
&= f(z_0) + (z - z_0)^k \left[\frac{1}{k!} f^k(z_0) + \frac{1}{(k+1)!}(z - z_0) f^{k+1}(z_0) + \cdots \right] \\
&= f(z_0) + (z - z_0)^k g(z) \text{ (say)}
\end{aligned}
$$

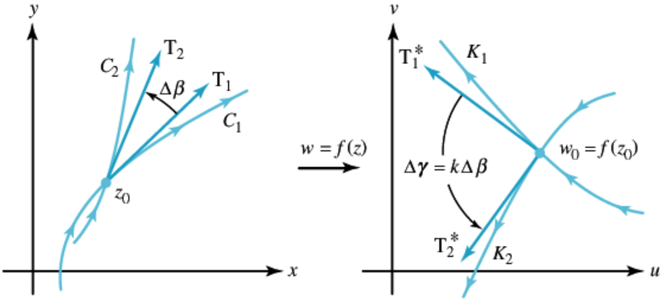
Scale Factors

- That means
$$f(z) - f(z_0) = (z - z_0)^k g(z)$$
and
$$\arg\,(w - w_0) = \arg\,(f(z) - f(z_0)) = k\,\arg\,(z - z_0) + \arg\,g(z).$$
- Let C_1, C_2 be two smooth curves passing through z_0.
- Let the image curves be $K_1 = f(C_1)$ and $K_2 = f(C_2)$.
- If z_1 is a variable points approaching to z_0 along C_1, then $w_1 = f(z_1)$ will approach to $w_0 = f(z_0)$ along the image curves K_1.
- Similarly if z_2 is a variable points approaching to z_0 along C_2, then $w_2 = f(z_2)$ will approach to $w_0 = f(z_0)$ along the image curves K_2
- Let θ and Φ be the angle between C_1, C_2 at z_0 and between K_1, K_2 at $f(z_0)$ respectively.

Scale Factors

Then

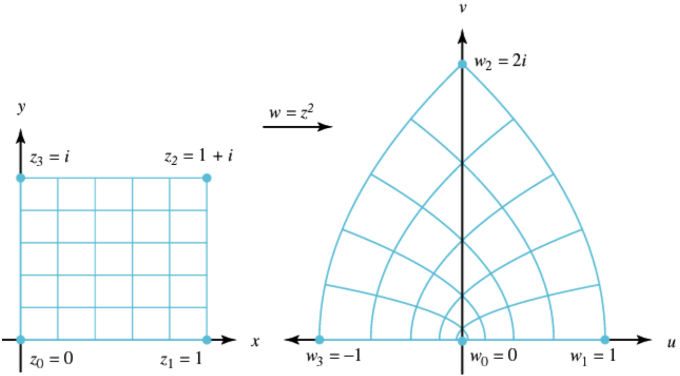
$$\begin{aligned}
\Phi &= \lim_{z_1, z_2 \to z_0} \arg\left(\frac{f(z_1) - f(z_0)}{f(z_2) - f(z_0)}\right) \\
&= \lim_{z_1, z_2 \to z_0} \arg\left(\frac{(z_1 - z_0)^k g(z_1)}{(z_2 - z_0)^k g(z_2)}\right) \\
&= \lim_{z_1, z_2 \to z_0} \left[k \arg \frac{z_1 - z_0}{z_2 - z_0} + \arg \frac{g(z_1)}{g(z_2)}\right] \\
&= k\theta.
\end{aligned}$$



Scale Factors

Question: Find the image of the unit square $S = \{x + iy : 0 < x < 1, 0 < y < 1\}$ under the map $w = f(z) = z^2$?

Answer: The map $f(z)$ is conformal at all $z \neq 0$. So the vertices $z_1 = 1, z_2 = 1 + i$ and $z_3 = i$ are mapped onto right angles at he vertices $w_1 = 1, w_2 = 2i$ and $w_3 = i$. But $f''(0) = 2 \neq 0$. so the angle at the vertex z_0 is magnified by the factor 2.



Möbius transformations

If a, b, c and d are complex constants such that $ad - bc \neq 0$, then the function

$$w = S(z) = \frac{az + b}{cz + d}$$

is called a Möbius transformation. It is also known as a bilinear transformation or a linear fractional transformation.

Observation:

- Every Möbius transformation is a conformal map.

$$S'(z) = \frac{(cz + d)a - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} \neq 0.$$

- The map S is analytic on $\mathbb{C} \setminus \{-\frac{d}{c}\}$.
- Composition of two Möbius transformation is a Möbius transformation.

- Define $T(w) = \dfrac{-dw + b}{cw - a}$ then
$$SoT(w) = w \;\; \text{and} \;\; ToS(z) = z.$$
- So S is invertible and $S^{-1} = T$.
- The map $S : \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$ is one one and onto.
- If we define $S(-\frac{d}{c}) = \infty$ and $S(\infty) = \dfrac{a}{c}$, then
$$S : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$
is a bijection.

Möbius transformations

Types of Möbius transformations:

- Let $a \in \mathbb{C}$. Then $S(z) = z + a$ is called *translation* map.
- Let $0 \neq a \in \mathbb{C}$. Then $S(z) = az$ is called *dilation* map. (If $|a| < 1$, S is called contraction map and if $|a| > 1$, S is called expansion map.)
- Let $\theta \in \mathbb{R}$. Then $S(z) = e^{i\theta} z$ is called *rotation* map.
- The map $S(z) = \frac{1}{z}$ is called *inversion* map.

Möbius transformations

- Every Möbius transformation is the composition of translations, dilations and the inversion.

Proof.

- Let $w = S(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ be a Möbius transformation. There are two possibilities:
- First, $c = 0$ in this case $S(z) = (a/d)z + (b/d)$. So if take

$$S_1(z) = (a/d)z, S_2(z) = z + (b/d),$$

then $S_2 \circ S_1 = S$ and we are done.

- Now if $c \neq 0$ in this case we take

$$S_1(z) = z + d/c, S_2(z) = 1/z, S_3(z) = \frac{bc - ad}{c^2} z,\ \ S_4(z) = z + a/c.$$

Then

$$S_4 \circ S_3 \circ S_2 \circ S_1(z) = \frac{az+b}{cz+d} = S(z).$$

Möbius transformations

- A point $z_0 \in \mathbb{C}_\infty$ is called a fixed point of a function f if $f(z_0) = z_0$.
- **Question:** What are the fixed point of a Möbius transformation S? That is what are the point z satisfying $S(z) = z$.
  Answer:
- If z satisfies the condition
  $$S(z) = \frac{az+b}{cz+d} = z,$$
  then
  $$cz^2 - (a-d)z - b = 0.$$
- A Möbius transformation can have at most two fixed points unless it is an identity map.
- If $c = 0$, then $z = -\frac{b}{a-d}$ (∞ if $a = d$) is the only fixed point of S .

Möbius transformations

Question: How many Möbius transformation are possible by its action on three distinct points in $\mathbb{C}_\infty$?

Answer: One!

Proof. Let S and T be two Möbius transformations such that

$$S(a) = T(a) = \alpha, \quad S(b) = T(b) = \beta \text{ and } \quad S(c) = T(c) = \gamma,$$

where a, b, c are three distinct points in $\mathbb{C}_\infty$. Consider

$$T^{-1} \circ S(a) = a, T^{-1} \circ S(b) = b \text{ and } T^{-1} \circ S(c) = c.$$

So we have a Möbius transformation $T^{-1} \circ S$ having three fixed points. Hence $T^{-1} \circ S = I$. That is $S = T$.

Question: How to find a Möbius transformation if its action on three distinct points in $\mathbb{C}_\infty$ is given?

Möbius transformations

Definition: Given four distinct points on the $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$, the cross ratio of z_1, z_2, z_3, z_4 is defined by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

- If $z_2 = \infty$ then $(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)}{(z_1 - z_4)}$
- If $z_3 = \infty$ then $(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_4)}{(z_1 - z_4)}$
- If $z_4 = \infty$ then $(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)}{(z_2 - z_3)}$
- The cross ratio is a Möbius transformation defined by

  $$S(z) = (z, z_2, z_3, z_4) = \frac{(z - z_3)(z_2 - z_4)}{(z_2 - z_3)(z - z_4)}$$

  such that $S(z_2) = 1, S(z_3) = 0$ and $S(z_4) = \infty$.

Möbius transformations

- Let T be any Möbius transformation. Then ST^{-1} is a Möbius transformation such that

$$ST^{-1}(Tz_2) = 1, \quad ST^{-1}(Tz_3) = 0 \text{ and } ST^{-1}(Tz_4) = \infty.$$

- Since a Möbius transformation is uniquely determined by its action on three distinct points in $\mathbb{C}_\infty$.
- So, we have

$$ST^{-1}(z) = (z, T(z_2), T(z_3), T(z_4)).$$

- In particular
$ST^{-1}(T(z_1)) = S(z_1) = (z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4)).$
Above discussion summarizes as:
- **Result:** If z_1, z_2, z_3, z_4 are four distinct points in $\mathbb{C}_\infty$ and T is any Möbius transformation then

$$(z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4)).$$

In other words, the cross ratio is invariant under Möbius transformation.

Result: If z_2, z_3, z_4 are three distinct points in $\mathbb{C}_\infty$ and if w_2, w_3, w_4 are also three distinct points of $\mathbb{C}_\infty$, then there is one and only one Möbius transformations S such that $Sz_2 = w_2, Sz_3 = w_3, Sz_4 = w_4$.

Proof

- Let $T(z) = (z, z_2, z_3, z_4)$, $M(z) = (z, w_2, w_3, w_4)$ and put $S = M^{-1} \circ T$.
- Clearly S has desire property.
- If R is another Möbius transformations. with $Rz_j = w_j$ for $j = 2, 3, 4$ then $R^{-1} \circ S$ has three fixed poins (z_2, z_3 and z_4).
- Hence $R^{-1} \circ S = I$ or $R = S$.

Möbius transformations

Question: Find a Möbius transformation that maps $z_1 = 1, z_2 = 0, z_3 = -1$ onto the points $w_1 = i, w_2 = \infty, w_3 = 1$.

Answer: We know that

$$(z, z_1, z_2, z_3) = (T(z), T(z_1), T(z_2), T(z_3)).$$

i.e.

$$(z, 1, 0, -1) = (T(z), i, \infty, 1)$$

which on solving gives

$$T(z) = \frac{(i+1)z + (i-1)}{2z}.$$

Möbius transformations

Theorem. A Möbius transformation takes circles onto circles.
Proof.

- Recall that every Möbius transformation is the composition of translations, dilations and the inversion.
- It is easy to show that translations, dilations takes circles onto circles.
- To prove this result it is enough to show that the transformation inversion takes circles onto circles.
- Consider the mapping $w = S(z) = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$.
- If $w = u + iv$ and $z = x + iy$ then

$$u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2}.$$

- Similarly,

$$x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2}.$$

Möbius transformations

- The general equation of a circle is

$$a(x^2 + y^2) + bx + cy + d = 0. \tag{1}$$

- Applying transformation $w = \frac{1}{z}$ (i.e. substituting $x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$) we have

$$a\left(\left(\frac{u}{u^2+v^2}\right)^2 + \left(-\frac{v}{u^2+v^2}\right)^2\right) + b\left(\frac{u}{u^2+v^2}\right) + c\left(-\frac{v}{u^2+v^2}\right) + d = 0$$

- Which on simplification becomes

$$d(u^2 + v^2) + bu - cv + a = o \tag{2}$$

Möbius transformations

- Find a Möbius transformation that takes UHP to RHP.
- Find the image of unit disc under the map $w = f(z) = \frac{z}{1-z}$.
  Ans: Re $w > -\frac{1}{2}$.
- Find the image of $D = \{z : |z+1| < 1\}$ under the map

$$w = f(z) = \frac{(1-i)z+2}{(1+i)z+2}.$$