Evaluation of integrals

Evaluation of certain contour integrals: Type I

Type I: Integrals of the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

- If we take $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ and $d\theta = \dfrac{dz}{iz}$.
- Substituting for $\sin\theta, \cos\theta$ and $d\theta$ the definite integral transforms into the following contour integral

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} f(z)\, dz$$

where $f(z) = \frac{1}{iz}[F(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z}))]$

- Apply Residue theorem to evaluate

$$\int_{|z|=1} f(z)\, dz.$$

Example of Type I

Consider

$$\int_0^{2\pi} \frac{1}{1+3(\cos t)^2}\, dt.$$

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+3(\cos t)^2}\, dt &= \int_{|z|=1} \frac{1}{1+3(\frac{1}{2}(z+\frac{1}{z}))^2}\,\frac{dz}{iz} \\
&= -4i \int_{|z|=1} \frac{z}{3z^4+10z^2+3}\, dz \\
&= -4i \int_{|z|=1} \frac{z}{3(z+\sqrt{3}i)\left(z-\sqrt{3}i\right)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)}\, dz \\
&= -\frac{4}{3} i \int_{|z|=1} \frac{z}{(z+\sqrt{3}i)(z-\sqrt{3}i)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)}\, dz \\
&= -\frac{4}{3} i \times 2\pi i \{Res(f, \frac{i}{\sqrt{3}}) + Res(f, -\frac{i}{\sqrt{3}})\}.
\end{aligned}$$

Improper Integrals of Rational Functions

- The improper integral of a continuous function f over $[0, \infty)$ is defined by

$$\int_0^\infty f(x)dx = \lim_{b\to\infty} \int_0^b f(x)dx$$

provided the limit exists.

- If f is defined for all real x, then the integral of f over $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^0 f(x)dx + \lim_{b\to\infty} \int_0^b f(x)dx$$

provided both limits exists.

- There is another value associated with the improper integral $\int_{-\infty}^{\infty} f(x)dx$ namely the **Cauchy Principal value(P.V.)** and it is given by

$$\text{P. V.} \int_{-\infty}^\infty f(x)dx := \lim_{R\to\infty} \int_{-R}^R f(x)dx$$

provided the limit exists.

Evaluation of certain contour integrals: Type II

- If the improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, then P. V. $\int_{-\infty}^{\infty} f(x)dx$ exists and
$$\int_{-\infty}^{\infty} f(x)dx = \text{P. V.} \int_{-\infty}^{\infty} f(x)dx.$$
- The P. V. $\int_{-\infty}^{\infty} f(x)dx$ exists $\not\Longrightarrow$ the improper integral $\int_{-\infty}^{\infty} f(x)dx$ exists. Take $f(x) = x$.
- However if f is an even function (i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$) then P. V. $\int_{-\infty}^{\infty} f(x)dx$ exists $\Longrightarrow$ the improper integral $\int_{-\infty}^{\infty} f(x)dx$ exists and their values are equal.

Consider the rational function $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials with real coefficients such that

- $Q(z)$ has no zeros in the real line
- degree of $Q(z) > 1+$ degree of $P(z)$

then P. V. $\int_{-\infty}^{\infty} f(x)dx$ can be evaluated using Cauchy residue theorem.

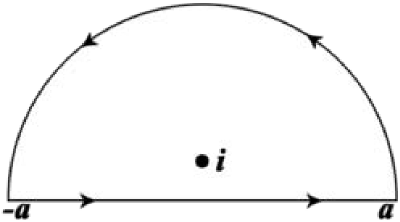
Type II Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx,$$

To evaluate this integral, we look at the complex-valued function

$$f(z) = \frac{1}{(z^2+1)^2}$$

which has singularities at i and $-i$. Consider the contour C like semicircle, the one shown below.



Note that:

$$\int_C f(z)\,dz = \int_{-a}^{a} f(z)\,dz + \int_{\text{Arc}} f(z)\,dz$$

$$\int_{-a}^{a} f(z)\,dz = \int_C f(z)\,dz - \int_{\text{Arc}} f(z)\,dz$$

Furthermore observe that

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}.$$

Then, by using Residue Theorem,

$$\int_C f(z)\,dz = \int_C \frac{\frac{1}{(z+i)^2}}{(z-i)^2}\,dz = 2\pi i \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\Bigg|_{z=i} = \frac{\pi}{2}$$

Evaluation of certain contour integrals: Type II

If we call the arc of the semicircle 'Arc', we need to show that the integral over 'Arc' tends to zero as a using the estimation lemma

$$\left| \int_{\text{Arc}} f(z)\, dz \right| \leq ML$$

where M is an upper bound on $|f(z)|$ along the Arc and L the length of 'Arc'. Now,

$$\left| \int_{\text{Arc}} f(z)\, dz \right| \leq \frac{a\pi}{(a^2-1)^2} \to 0 \text{ as } a \to \infty.$$

So

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = \int_{-\infty}^{\infty} f(z)\, dz = \lim_{a\to+\infty} \int_{-a}^{a} f(z)\, dz = \frac{\pi}{2}.$$

Type III Integrals of the form

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \, dx \text{ or P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \, dx,$$

where

- $P(x), Q(x)$ are real polynomials and $m > 0$
- $Q(x)$ has no zeros in the real line
- degree of $Q(x) >$ degree of $P(x)$

then

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \, dx \text{ or P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \, dx$$

can be evaluated using Cauchy residue theorem.

Evaluation of certain contour integrals: Type III

- **Jordan's Lemma:** If $0 < \theta \leq \frac{\pi}{2}$ then $\frac{2\theta}{\pi} \leq \sin\theta \leq \theta$.

  Proof: Define $\phi(\theta) = \frac{\sin\theta}{\theta}$. Then $\phi'(\theta) = \frac{\psi(\theta)}{\theta^2}$, where $\psi(\theta) = \theta\cos\theta - \sin\theta$.

  1. Since $\psi(0) = 0$ and $\psi'(\theta) = -\theta\sin\theta \leq 0$ for $0 < \theta \leq \frac{\pi}{2}$, ψ decreases as θ increases i.e. $\psi(\theta) \leq \psi(0) = 0$ for $0 < \theta \leq \frac{\pi}{2}$.
  2. So $\phi'(\theta) = \frac{\psi(\theta)}{\theta^2} \leq 0$ for $0 < \theta \leq \frac{\pi}{2}$.
  3. That means ϕ is decreasing and hence $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$ for $0 < \theta \leq \frac{\pi}{2}$.

- By Jordan's lemma we have

$$\begin{aligned}
\int_0^{\pi} e^{-a\sin\theta} d\theta &= \int_0^{\frac{\pi}{2}} e^{-a\sin\theta} d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-a\sin\theta} d\theta \\
&\leq \int_0^{\frac{\pi}{2}} e^{-a\frac{2\theta}{\pi}} d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-a\frac{2(\pi-\theta)}{\pi}} d\theta
\end{aligned}$$

both the integrals goes to 0 as $a \to \infty$.

Evaluation of certain contour integrals: Type III

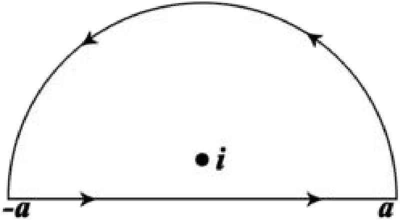
Evaluate:

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+1}\,dx \text{ or } \int_{-\infty}^{\infty} \frac{\sin tx}{x^2+1}\,dx$$

Consider the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$

We will evaluate it by expressing it as a limit of contour integrals along the contour C that goes along the real line from $-a$ to a and then counterclockwise along a semicircle centered at 0 from a to $-a$. Take $a > 1$ to be greater than 1, so that i is enclosed within the curve.



$$\operatorname{Res}\left(\frac{e^{itz}}{z^2+1}, i\right) = \lim_{z\to i}(z-i)\frac{e^{itz}}{z^2+1} = \lim_{z\to i}\frac{e^{itz}}{z+i} = \frac{e^{-t}}{2i}.$$

So by residue theorem

$$\int_C f(z)\,dz = (2\pi i)\operatorname{Res}_{z=i} f(z) = 2\pi i\frac{e^{-t}}{2i} = \pi e^{-t}.$$

The contour C may be split into a "straight" part and a curved arc, so that

$$\int_{\text{straight}} + \int_{\text{arc}} = \pi e^{-t}$$

and thus

$$\int_{-a}^{a}\frac{e^{itx}}{x^2+1}\,dx = \pi e^{-t} - \int_{\text{arc}}\frac{e^{itz}}{z^2+1}\,dz.$$

Evaluation of certain contour integrals: Type III

$$\begin{aligned}\left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz\right| &\leq \int_0^{\pi} \left| a\frac{e^{ita(cos\theta + i\sin\theta)}}{a^2-1}\right| d\theta \\ &\leq \frac{a}{a^2-1}\int_0^{\pi} e^{-ta\sin\theta}\,d\theta.\end{aligned}$$

Hence,

$$\left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz\right| \to 0 \text{ as } a \to \infty$$

and

$$\begin{aligned}\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos tx}{x^2+1}\,dx &= \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{itx}}{x^2+1}\,dx \\ &= \lim_{a\to\infty} \left[\pi e^{-t} - \int_{\text{arc}} \frac{e^{itz}}{z^2+1}\,dz\right] \\ &= \pi e^{-t}.\end{aligned}$$

Type IV Integrals of the form

$$\int_0^\infty \frac{\sin x}{x} dx$$

can be evaluated using Cauchy residue theorem.
Before we discuss integrals of Type IV we need the following result.
Lemma: Suppose f has a simple pole at $z = a$ on the real axis. If c_ρ is the contour defined by $c_\rho(t) = a + \rho e^{i(\pi - t)},\ t \in (0, \pi)$ then

$$\lim_{\rho \to 0} \int_{c_\rho} f(z)dz = -i\pi \text{Res}(f, a).$$

Proof: Since f has a simple pole at $z = a$, the Laurent series expansion of f about $z = a$ is of the form

$$f(z) = \frac{\text{Res}(f, a)}{z - a} + g(z).$$

Evaluation of certain contour integrals: Type IV

Now

$$\begin{aligned}\int_{c_\rho} f(z)dz &= \int_{c_\rho} \frac{\text{Res}(f,a)}{z-a}dz + \int_{c_\rho} g(z)dz \\ &= -\text{Res}(f,a)\int_0^\pi \frac{i\rho e^{i(\pi-t)}}{\rho e^{i(\pi-t)}}dt - \int_0^\pi g(a+\rho e^{i(\pi-t)})i\rho e^{i(\pi-t)}dt \\ &= -i\pi\text{Res}(f,a) - \int_0^\pi g(a+\rho e^{i(\pi-t)})i\rho e^{i(\pi-t)}dt.\end{aligned}$$

Note that f has Laurent series expansion in $0 < |z-a| < R$ for some $R > 0$. The function g is continuous on $|z-a| \leq \rho_0$ for every $\rho < \rho_0 < R$. So $|g(z)| < M$ on $|z-a| \leq \rho_0$. So

$$\left|\int_0^\pi g(a+\rho e^{i(\pi-t)})i\rho e^{i(\pi-t)}dt\right| \leq \rho M\pi \to 0 \text{ as } \rho \to 0.$$

Hence

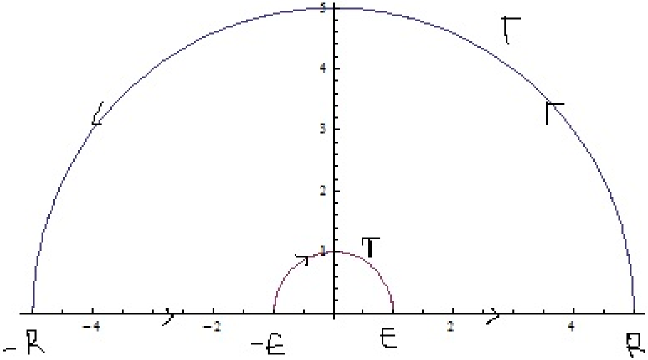
$$\lim_{\rho\to 0}\int_{c_\rho} f(z)dz = -i\pi\text{Res}(f,a).$$

Consider the integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

Define $f(z) = \frac{e^{iz}}{z}$, ($z = 0$ is a simple pole on the real axis).

Consider the contour $C = [-R, -\epsilon] \cup \tau \cup [\epsilon, R] \cup \Gamma$.



By Cauchy's theorem

$$\int_C \frac{e^{iz}}{z}dz = \int_{[-R,-\epsilon]} \frac{e^{iz}}{z}dz + \int_\tau \frac{e^{iz}}{z}dz + \int_{[\epsilon,R]} \frac{e^{iz}}{z}dz + \int_\Gamma \frac{e^{iz}}{z}dz = 0.$$

But

$$\int_{[-R,-\epsilon]} \frac{e^{iz}}{z}dz + \int_{[\epsilon,R]} \frac{e^{iz}}{z}dz = \int_{[\epsilon,R]} \frac{e^{ix} - e^{-ix}}{x}dx$$

So

$$\int_{[\epsilon,R]} \frac{e^{ix} - e^{-ix}}{x}dx = -\int_\tau \frac{e^{iz}}{z}dz - \int_\Gamma \frac{e^{iz}}{z}dz = i\pi$$

as $\epsilon \to 0$ (by the previous Lemma) and $R \to \infty$ (by Jordan's inequality) and hence,

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

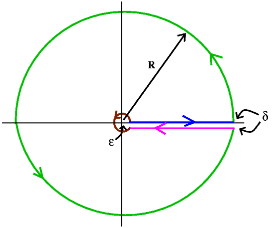
Integration along a branch cut: Consider the improper integral

$$\int_0^\infty \frac{x^{-a}}{1+x}dx \quad (0 < a < 1).$$

Define

$$f(z) = \frac{z^{-a}}{1+z} \quad (|z| > 0,\ 0 < \arg z < 2\pi).$$

- The function $\frac{z^{-a}}{1+z}$ is a multiple valued function with branch cut $\arg z = 0$ (positive real axis).
- Consider the contour $C = [\epsilon + i\delta, R + i\delta] \cup \Gamma_R \cup [R - i\delta, \epsilon - i\delta] \cup \{-\gamma_\epsilon\}$.



Evaluation of certain contour integrals: Type V

By residue theorem

$$\left(\int_{[\epsilon+i\delta,R+i\delta]} + \int_{\Gamma_R} + \int_{[R-i\delta,\epsilon-i\delta]} + \int_{-\gamma_\epsilon}\right) f(z)dz = 2\pi i \mathrm{Res}(f,-1) = 2\pi i e^{-ia\pi}.$$

Since

$$f(z) = \frac{exp(-a\log z)}{z+1} = \frac{exp(-a(\ln r + i\theta))}{re^{i\theta}+1},$$

where $z = re^{i\theta}$, it follows that

On $[\epsilon + i\delta, R + i\delta]$, $\theta \to 0$ as $\delta \to 0$,

$$f(z) = \frac{exp(-a(\ln r + i.0))}{re^{i.0}+1} \to \frac{r^{-a}}{1+r} \quad \text{as} \quad \delta \to 0.$$

On $[R - i\delta, \epsilon - i\delta]$, $\theta \to 2\pi$ as $\delta \to 0$,

$$f(z) = \frac{exp(-a(\ln r + i.2\pi))}{re^{i.2\pi}+1} \to \frac{r^{-a}}{1+r}e^{-2a\pi i} \quad \text{as} \quad \delta \to 0.$$

But

$$\left|\int_{\Gamma_R} \frac{z^{-a}}{1+z}dz\right| \le \frac{R^{-a}}{R-1}2\pi R = \frac{2\pi R}{R-1}\frac{1}{R^a} \to 0 \text{ as } R \to \infty$$

and

$$\left|\int_{\gamma_\epsilon} \frac{z^{-a}}{1+z}dz\right| \le \frac{\epsilon^{-a}}{\epsilon-1}2\pi\epsilon = \frac{2\pi}{1-\epsilon}\epsilon^{1-a} \to 0 \text{ as } \epsilon \to 0.$$

So

$$\lim_{R\to\infty,\epsilon\to 0}\left(\int_\epsilon^R \frac{r^{-a}}{1+r}dr + \int_R^\epsilon \frac{r^{-a}}{1+r}e^{-2a\pi i}dr\right) = 2\pi i e^{-ia\pi}$$

That is

$$(1-e^{-2a\pi i})\int_0^\infty \frac{r^{-a}}{1+r}dr = 2\pi i e^{-ia\pi}$$

and hence

$$\int_0^\infty \frac{r^{-a}}{1+r}dr = \frac{2\pi i e^{-ia\pi}}{(1-e^{-2a\pi i})} = \frac{\pi}{\sin a\pi} \quad (0 < a < 1).$$

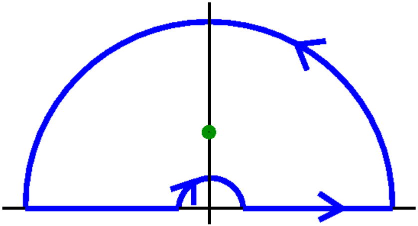
Integration around a branch cut:
Consider the improper integral

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx.$$

Define

$$f(z) = \frac{\log z}{1+z^2} \quad \left(|z|>0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right).$$

- The function $\dfrac{\log z}{1+z^2}$ is a multiple valued function whose branch cut consists of origin and negative imaginary axis.
- Consider the contour $C = [\epsilon, R] \cup \Gamma_R \cup [-R, -\epsilon] \cup \{-\gamma_\epsilon\}$.



Evaluation of certain contour integrals: Type VI

By Cauchy's residue theorem

$$\left(\int_{[\epsilon,R]} + \int_{\Gamma_R} + \int_{[-R,-\epsilon]} + \int_{-\gamma_\epsilon}\right) f(z)dz = 2\pi i \text{Res}(f,i) = 2\pi i \frac{\pi}{4} = \frac{\pi^2 i}{2}.$$

Since

$$f(z) = \frac{\log z}{z^2+1} = \frac{\log|z| + i\theta}{r^2 e^{2i\theta}+1},$$

where $z = re^{i\theta}$, it follows that

On $[\epsilon, R]$, $\theta = 0$,

$$f(z) = \frac{\log x}{x^2+1}.$$

On $[-R, -\epsilon]$, $\theta = \pi$,

$$f(z) = \frac{\log|x| + i\pi}{x^2+1}.$$

Evaluation of certain contour integrals: Type VI

But

$$\left|\int_{\Gamma_R} \frac{\log z}{1+z^2} dz\right| = \left|\int_{\Gamma_R} \frac{\log R + i\theta}{1+R^2 e^{2i\theta}} iRe^{i\theta} d\theta\right|$$

$$\leq R\frac{|\log R|}{R^2-1}\pi + \frac{R}{R^2-1}\int_0^{\pi} \theta d\theta \to 0$$

as $R \to \infty$ and

$$\left|\int_{\gamma_\epsilon} \frac{\log z}{1+z^2} dz\right| = \left|\int_{\gamma_\epsilon} \frac{\log \epsilon + i\theta}{1+\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta} d\theta\right|$$

$$\leq \epsilon\pi\frac{|\log \epsilon|}{\epsilon^2-1} + \frac{\epsilon}{\epsilon^2-1}\int_0^{\pi} \theta d\theta \to 0 \text{ as } \epsilon \to 0.$$

So

$$\lim_{R\to\infty,\epsilon\to 0}\left(\int_{\epsilon}^{R}\frac{\log x}{x^2+1}dx+\int_{-R}^{-\epsilon}\frac{\log|x|+i\pi}{x^2+1}dx\right)=\frac{\pi^2 i}{2}$$

That is

$$\lim_{R\to\infty,\epsilon\to 0}\left(\int_{\epsilon}^{R}\frac{\log x}{x^2+1}dx+\int_{\epsilon}^{R}\frac{\log|x|}{x^2+1}dx+\int_{\epsilon}^{R}\frac{i\pi}{x^2+1}dx\right)=\frac{\pi^2 i}{2}.$$

Hence

$$\int_0^{\infty}\frac{\log x}{x^2+1}dx=0$$

and

$$\int_0^{\infty}\frac{1}{x^2+1}dx=\frac{\pi}{2}.$$