MA 201 Complex Analysis
Lecture 14: Laurent Series and Singularities

Laurent's Series

Suppose that $0 \leq r < R$. Let f be an analytic defined on the annulus

$$A = \text{ann}(a, r, R) = \{z : r < |z - a| < R\}.$$

- Then for each $z \in A, f(z)$ has the Laurrent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$$

- where the convergence is absolute and uniform in $\overline{\text{ann}(a, r_1, R_1)}$ if $r < r_1 < R_1 < R$.
- The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}}\, dz$$

  where $\gamma(t) = a + se^{it},\ t \in [0, 2\pi]$ and for any $r < s < R$.
- Moreover, this series is unique and $\displaystyle\sum_{k=-\infty}^{-1} a_n (z - a)^n$ is called **(principal part)** and $\displaystyle\sum_{k=0}^{\infty} a_n (z - a)^n$ is called **(regular/analytic part)**.

Examples of Laurent Series

Let $f(z) = \dfrac{1}{1-z}$.

- On the domain $|z| < 1$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots + .$$

- On the domain $|z| > 1$ i.e. $\frac{1}{|z|} < 1$, by the above mentioned fact we have

$$f(z) = \frac{1}{1-z} = \frac{-1}{z(1-\frac{1}{z})} = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = -\frac{1}{z} - \frac{1}{z^2} - \ldots .$$

Examples of Laurent Series

- Let $f(z) = \frac{\sin z}{z}$ domain $|z| > 0$

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{6} + \frac{z^4}{120} + \cdots .$$

- Let $f(z) = \frac{e^z - 1}{z^3}$ domain $|z| > 0$

$$f(z) = \frac{e^z - 1}{z^3} = \frac{1}{z^3}\sum_{1}^{\infty} \frac{z^n}{n!} = \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \frac{z}{24} + \cdots .$$

- Let $f(z) = e^{\frac{1}{z}}$ domain $|z| > 0$

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots .$$

Examples of Laurent Series

Let $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

- Domain $|z| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (1 - 2^{-n-1}) z^n.$$

- Domain $1 < |z| < 2$ i.e. $\dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{2} < 1$, so we have

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

- Domain $|z| > 2$ i.e. $\dfrac{1}{|z|} < 1$ and $\dfrac{2}{|z|} < 1$, in this case we have

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{2^{n-1} - 1}{z^n}.$$

Singularities

Behavior of following functions f at 0:

- $f(z) = \dfrac{1}{z^9}$
- $f(z) = \dfrac{\sin z}{z}$
- $f(z) = \dfrac{e^z - 1}{z}$
- $f(z) = \dfrac{1}{\sin(\frac{1}{z})}$
- $f(z) = \text{Log } z$
- $f(z) = e^{\frac{1}{z}}$

In the above we observe that all the functions are not analytic at 0, however in every neighborhood of 0 there is a point at which f is analytic.

Singularities

- **Definition:** The point z_0 is called a **singular point** or **singularity of f**

  1. if f is not analytic at z_0 but
  2. every neighborhood of z_0 contains at least one point at which f is analytic.

- **Examples:** $\dfrac{e^z - 1}{z}, \dfrac{1}{z^2}, \sin\frac{1}{z}, \text{Log } z$ etc. has singularity at $z = 0$.

- **Note:** $\bar{z}, |z|^2, \text{Re } z, \text{Im } z, z\text{Re } z$ are nowhere analytic. That does not mean that every point of $\mathbb{C}$ is a singularity.

Singularities

- A singularities are classified into TWO types:
  1. A singular point z_0 is said to be an **isolated singularity or isolated singular point** of f if f is analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$.
  2. A singular point z_0 is said to be an **non-isolated singularity** if z_0 is not an isolated singular point.
- $\dfrac{\sin z}{z}, \dfrac{1}{z^2}, \sin(\frac{1}{z})$ (0 is isolated singular point).
- $\dfrac{1}{\sin(\pi/z)}, Log\ z$ these functions has non-isolated singularity at 0.

Singularities

If f has an isolated singularity at z_0, then f is analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$. In this case f has the following Laurent series expansion:

$$f(z) = \cdots \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - a) + a_2(z - z_0)^2 + \cdots .$$

- If all $a_{-n} = 0$ for all $n \in \mathbb{N}$, then the point $z = z_0$ is a **removal singularity**.
- The point $z = z_0$ is called a **pole** if all but a finite number of a_{-n}'s are non-zero. If m is the highest integer such that $a_{-m} \neq 0$, then z_0 is a Pole of order m.
- If $a_{-n} \neq 0$ for infinitely many n's, then the point $z = z_0$ is a **essential singularity**.
- The term a_{-1} is called **residue** of f at z_0.

Removable singularities

- The following statements are equivalent:

  1. f has a removable singularity at z_0.
  2. If all $a_{-n} = 0$ for all $n \in \mathbb{N}$.
  3. $\lim_{z \to z_0} f(z)$ exists and finite.
  4. $\lim_{z \to z_0} (z - z_0) f(z) = 0$.
  5. f is bounded in a deleted neighborhood of z_0.

- The function $\dfrac{\sin z}{z}$ has removable singularity at 0.

Pole

The following statements are equivalent:

1. f has a pole of order m at z_0.
2. $f(z) = \frac{g(z)}{(z - z_0)^m}$, g is analytic at z_0 and $g(z_0) \neq 0$.
3. $\frac{1}{f}$ has a zero of order m.
4. $\lim_{z \to z_0} |f(z)| = \infty$.
5. $\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$
6. $\lim_{z \to z_0} (z - z_0)^m f(z)$ has removal singularity at z_0.

Examples: $\frac{e^z - 1}{z^8}$ has pole of order 7.

Essential singularity

The following statements are equivalent:

- f has a essential singularity at z_0.
- The point z_0 is neither a pole nor removable singularity.
- $\lim_{z \to z_0} f(z)$ does not exists.
- Infinitely many terms in the principal part of Laurent series expansion around the point z_0.

Limit point of zeros is isolated essential singularity. For example:

$$f(z) = \sin \frac{1}{z}$$

Singularities at ∞

Let f be a complex valued function. Define another function g by

$$g(z) = f\left(\frac{1}{z}\right).$$

Then the nature of singularity of f at $z = \infty$ is defined to be the the nature of singularity of g at $z = 0$.

- $f(z) = z^3$ has a pole of order 3 at ∞.
- e^z has an essential singularity at ∞.
- An entire function f has a removal singularity at ∞ if and only if f is constant.(Prove This!)
- An entire function f has a pole of order m at ∞ if and only if f is a polynomial of degree m.(Prove This!)