# MA 201 Complex Analysis
# Lecture 13:
# Identity Theorem and Maximum Modulus Theorem

# Zeros of analytic functions

Suppose that $f : D \to \mathbb{C}$ is analytic on an open set $D \subset \mathbb{C}$.

- A point $z_0 \in D$ is called zero of $f$ if $f(z_0) = 0$.
- The $z_0$ is a zero of multiplicity/order $m$ if there is an analytic function $g : D \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^m g(z), \;\; g(z_0) \neq 0.$$

- In this case $f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ but $f^m(z_0) \neq 0$.

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# Zeros of analytic functions

- **Understanding of multiplicity via Taylor's series:** If $f$ is analytic function in $D$, then $f$ has a Taylor series expansion around $z_0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n, \quad |z - z_0| < R.$$

- If $f$ has a zero of order $m$ at $z_0$ then

$$f(z) = (z - z_0)^m \sum_{n=m}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^{n-m}$$

- Define $g(z) = \sum_{n=m}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^{n-m}$, then

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# Zeros of analytic functions

**Zeros of a non-constant analytic function are isolated:** If $f : D \to \mathbb{C}$ is non-constant and analytic at $z_0 \in D$ with $f(z_0) = 0$, then there is an $R > 0$ such that $f(z) \neq 0$ for $z \in B(z_0, R) \setminus \{z_0\}$.

**Proof.**

- Assume that $f$ has a zero at $z_0$ of order $m$. Then
$$f(z) = (z - z_0)^m g(z)$$
where $g(z)$ is analytic and $g(z_0) \neq 0$.
- Since $g$ is continuous at $z_0$ thus for $\epsilon = \dfrac{|g(z_0)|}{2} > 0$, we can find a $\delta > 0$ such that
$$|g(z) - g(z_0)| < \frac{|g(z_0)|}{2},$$
whenever $|z - z_0| < \delta$ .
- Therefore whenever $|z - z_0| < \delta$, we have $0 < \dfrac{|g(z_0)|}{2} < |g(z)| < \dfrac{3|g(z_0)|}{2}$. Take $R = \delta$.

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**Identity Theorem:** Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists an infinite sequence $\{z_k\} \subset D$, such that $f(z_k) = 0$, $\forall k \in \mathbb{N}$ and $z_k \to z_0 \in D$, $f(z) = 0$ for all $z \in D$.

**Proof.**

- **Case I:** If $D = \{z \in \mathbb{C} : |z - z_0| < r\}$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ for all } z \in D.$$

- We will show that $f^n(z_0) = 0$ for all $n$. If possible assume that $f^n(z_0) \neq 0$ for some $n > 0$.
- Let $n_0$ be the smallest positive integer such that $f^{n_0}(z_0) \neq 0$. Then

$$f(z) = \sum_{n=n_0}^{\infty} a_n (z - z_0)^n = (z - z_0)^{n_0} g(z),$$

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- Since $g$ is continuous at $z_0$, there exist $\epsilon > 0$ such that $g(z) \neq 0$ for all $z \in B(z_0, \epsilon)$.
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- **Case II:** If $D$ is a domain.
- Since $z_0 \in D$ therefore there exists $\delta > 0$ such that $B(z_0, \delta) \subset D$.
- By Case I, $f(z) = 0, \ \forall \ z \in B(z_0, \delta)$.
- Now take $z \in D$ join $z$ and $z_0$ by a line segment. Cover the line segments by open balls in such a way that center of a ball lies in the previous ball. Apply the above argument to get $f(z) = 0$ for all $z \in D$.

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# Uniqueness Theorem

**Uniqueness Theorem:** Let $D \subset \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ is analytic. If there exists an infinite sequence $\{z_n\} \subset D$, such that $f(z_n) = g(z_n)$, $\forall n \in \mathbb{N}$ and $z_n \to z_0 \in D$, $f(z) = g(z)$ for all $z \in D$.

- Find all entire functions $f$ such that $f(r) = 0$ for all $r \in Q$.
- Find all entire functions $f$ such that $f(x) = \cos x + i \sin x$ for all $x \in (0, 1)$.
- Find all analytic functions $f : B(0, 1) \to \mathbb{C}$ such that $f(\frac{1}{n}) = \sin(\frac{1}{n})$, $\forall n \in \mathbb{N}$.
- There does not exists an analytic function $f$ defined on $B(0, 1)$ such that $f(x) = |x|^3$ for all $x \in (-1, 1)$?

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# Maximum Modulus Theorem

**Maximum Modulus Theorem:** Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists a point $z_0 \in D$, such that $|f(z)| \leq |f(z_0)|$, $\forall z \in D$, then $f$ is constant on $D$.

**Proof.** Choose a $r > 0$ such that $\overline{B(z_0, r)} \subset D$. Let $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$. By Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Hence

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|\, dt \leq |f(z_0)|.$$

This gives

$$\int_0^{2\pi} \Big[ |f(z_0)| - |f(z_0 + re^{it})| \Big]\, dt = 0.$$

It follows that $|f(z_0)| = |f(z_0 + re^{it})|$ for all $t$. Now $f$ analytic and $|f|$ is constant gives $f$ is constant on $B(z_0, r)$. Applying identity theorem we get $f$ is constant through out the domain $D$.

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$$\int_0^{2\pi} \Big[ |f(z_0)| - |f(z_0 + re^{it})| \Big]\, dt = 0.$$

It follows that $|f(z_0)| = |f(z_0 + re^{it})|$ for all $t$. Now $f$ analytic and $|f|$ is constant gives $f$ is constant on $B(z_0, r)$. Applying identity theorem we get $f$ is constant through out the domain $D$.

# Maximum Modulus Theorem

**Maximum Modulus Theorem:** Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists a point $z_0 \in D$, such that $|f(z)| \leq |f(z_0)|, \ \forall z \in D$, then $f$ is constant on $D$.

**Proof.** Choose a $r > 0$ such that $\overline{B(z_0, r)} \subset D$. Let $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$. By Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Hence

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# Consequences of Maximum Modulus Theorem

- If $f$ is analytic in a bounded domain $D$ and continuous on $\partial D$ then $|f(z)|$ attains its maximum at some point on the boundary $\partial D$.

- Define $f(z) = e^{e^z}$ for $z \in D = \{z \in \mathbb{C} : |\text{Im } z| < \frac{\pi}{2}\}$. Then for $a + ib \in \partial D = \{\zeta \in \mathbb{C} : |\text{Im } \zeta| = \frac{\pi}{2}\}$,

$$f(a+ib) = \left| e^{e^{a \pm i\frac{\pi}{2}}} \right| = \left| e^{\pm i e^a} \right| = 1.$$

  Again if $x \in \mathbb{R} \subset D$ then, $f(x) = e^{e^x} \to \infty$ as $x \to \infty$.

- **Minimum Modulus Theorem** Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists a point $z_0 \in D$, such that $|f(z)| \geq |f(z_0)|$ for all $z \in D$, then either $f$ is constant function or $f(z_0) = 0$.

Hint. Apply maximum modulus theorem on $1/f$.

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