MA 201 Complex Analysis
Lecture 12: Taylor's Theorem

Taylor Series

- **Question:** Let $f : B(z_0, R) \to \mathbb{C}$ analytic. Can we represent f as a power series around z_0?
- **Taylor's Theorem:** Let f be analytic on $D = B(z_0, R)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{for all } z \in D,$$

where $a_n = \dfrac{f^n(z_0)}{n!}$ for $n = 0, 1, 2, \ldots.$

- **Proof.** Without loss of generality we consider $z_0 = 0$. Fix $z \in B(z_0, R)$ and let $|z| = r$. Let C_0 be a circle with center 0 and radius r_0 such that $r < r_0 < R$. Thus for two complex numbers w and z we can write (check!)

$$\frac{1}{w-z} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots. + \frac{z^{n-1}}{w^n} + \frac{z^n}{(w-z)w^n}.$$

Taylor Series

- By Cauchy integral formula we now have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|w|=r_0} \frac{f(w)dw}{w-z} \\ &= \frac{1}{2\pi i}\int_{|w|=r_0} f(w)\left[\frac{1}{w} + \frac{z}{w^2} + \ldots + \frac{z^{n-1}}{w^n}\right]dw + \frac{z^n}{2\pi i}\int_{|w|=r_0} \frac{f(w)dw}{(w-z)w^n} \\ &= f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \ldots.. + \frac{f^{n-1}(0)}{(n-1)!}z^{n-1} + \rho_n(z) \\ &= \sum_{k=0}^{n-1} \frac{f^k(0)}{k!}z^k + \rho_n(z) \end{aligned}$$

where $\rho_n(z) = \frac{z^n}{2\pi i}\int_{C_0} \frac{f(w)dw}{(w-z)w^n}$.

- Now, we just need to show that $\lim_{n\to\infty} |\rho_n(z)| = 0$.

- Note that the function $w \to \dfrac{f(w)}{w-z}$ is continuous and hence bounded on the circle C_0.

- If $\left|\dfrac{f(w)}{w-z}\right| \leq K$ for all $w \in C_0$ then by ML inequality it follows that

$$|\rho_n(z)| \leq K r_0 \left|\frac{z}{r_0}\right|^n.$$

- Since $|z| = r < r_0$ and $|w| = r_0$ implies $\lim_{n\to\infty} |\rho_n(z)| = 0$ as $n \to \infty$.

- **Remark:** If f is an entire function then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all } z \in \mathbb{C},$$

where $a_n = \dfrac{f^n(0)}{n!}$ for $n = 0, 1, 2, \ldots$.

Exponential function

The power series $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ has radius of convergence ∞. If we define $f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ then f is an entire function.

1. $f'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = f(z)$.
2. We know that $\frac{d}{dz} e^z = e^z$. Is $f(z) = e^z$?
3. **Yes.** If $h(z) = \frac{f(z)}{e^z}$ then $h'(z) = 0$ for all $z \in \mathbb{C}$. Therefore $f(z) = ce^z$. But $f(0) = e^0 = 1 = c$.

Now we will define exponential function as a power series

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Sine and Cosine function

The $\sin z$ and $\cos z$ functions can also be written as a power series by using the exponential series.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!},$$

similarly,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}.$$

Exponential function

Euler's Formula:

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(2n)!}(i\theta)^{2n} + \frac{1}{(2n+1)!}(i\theta)^{2n+1} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{\theta^{2n}(i^2)^n}{(2n!)} + i\frac{\theta^{2n+1}(i^2)^n}{(2n+1)!} \right] \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

Polynomial function

- If $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ is a polynomial in $\mathbb{C}$ of degree $n \geq 1$ then $|p(z)| \leq M|z|^n$ for $|z| > 1$.

- **Question:** Let f is an entire function such that $|f(z)| \leq M|z|^n$ for $|z| > 1$. Can we say that f is a polynomial of degree at most n?

- **Yes!** Since f is an entire function, therefore $f(z) = \sum a_k z^k$.

- By Cauchy's estimate,

$$|f^k(0)| \leq \frac{k!MR^n}{R^k} = \frac{k!M}{R^{(k-n)}} \to 0$$

  as $R \to \infty$ for each $k > n$.

- By above observation $a_k = 0$ for all $k > n$. i.e f is a polynomial of degree at most n.