MA 201 Complex Analysis
Lecture 11: Power Series

Series of complex numbers

If $z_n \in \mathbb{C}$ for all $n \geq 0$ then the series $\displaystyle\sum_{n=0}^{\infty} z_n$ converges

- to z if for every $\epsilon > 0$ there exists an positive integer N such that $\left|\displaystyle\sum_{n=0}^{m} z_n - z\right| < \epsilon$ for all $m \geq N$
- **absolutely** if $\sum |z_n|$ converges.
- If the series $\sum z_n$ converges absolutely then $\sum z_n$ converges.
- Let $S_N = \displaystyle\sum_{n=0}^{N} z_n$ be the Nth partial sum of $\sum z_n$. Then the series $\sum z_n$ converges if and only if the sequence $\{S_N\}$ converges.

Power Series

- Definition: A series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $a_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$ is called a **power series** around the point z_0.
- For what values of z the following power series converges?

1. $\sum_{n=0}^{\infty} z^n$ ($|z| < 1$, Geometric series.)
2. $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ (for all z, Exponential series.)
3. $\sum_{n=0}^{\infty} n! z^n$, (only $z = 0$)

Power Series

- If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \in \mathbb{C}$ then it converges for all $z \in \mathbb{C}$ such that $|z| < |z_0|$.
- **Proof.** It follows from the hypothesis that there exist $M \geq 0$ such that $|a_n z_0^n| \leq M$ for all $n \in \mathbb{N}$.
- Note that
$$|a_n z^n| = |a_n z_0|^n \left|\frac{z}{z_0}\right|^n \leq M \left|\frac{z}{z_0}\right|^n.$$
- The proof now follows from the comparison test, and behavior of geometric series.

Power Series

- (**Radius of convergence**)Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:
  1. If $|z| < R$ the series converges absolutely.
  2. If $|z| > R$ the series diverges.

  The number R is called the **radius of convergence** of a power series.

- a) For $\sum_{n=0}^{\infty} n! z^n$, $R = 0$. b) For $\sum_{n=0}^{\infty} z^n$, $R = 1$. c) For $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R = 1$. d) For $\sum_{n=1}^{\infty} \frac{z^n}{n!}$, $R = \infty$.
- **Remark:** Note that no conclusion about convergence can be drawn if $|z| = R$. The power series in c) above does not converge if $z = 1$ but converges if $z = -1$.

- The formula for calculating R goes exactly as in the case of reals, that is,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

whenever the above limits exist (with the supposition that division by ∞ (resp. 0) produces 0 (resp. ∞)).

- Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Then for all $z \in B(0, R)$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a well defined function.

- **Question:** Is f analytic on $B(0, R)$?

Theorem: Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R > 0$.

1. Then the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has radius of convergence R.
2. The function F is differentiable on $B(0, R)$ and furthermore $F'(z)$ is given by $F'(z) = f(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$

Proof. We know $\lim_{n\to\infty} n^{1/n} = 1$, and therefore

$$\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n},$$

so that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Now let $|z| < r < R$, write

$$F(z) = S_N(z) + E_N(z),$$

where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then if h is chosen so that $|z+h| < r$ we have

$$\begin{aligned} \frac{F(z+h)-F(z)}{h} - f(z) \quad &= \quad \left(\frac{S_N(z+h)-S_N(z)}{h} - S_N'(z)\right) \\ &+ \quad (S_N'(z) - f(z)) + \left(\frac{E_N(z+h)-E_N(z)}{h}\right). \end{aligned}$$

Now we will show that all three expression on right in the above equation will go to zero for large N and small h.

Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$, we see that

$$\left|\frac{E_N(z+h) - E_N(z)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z-h)^n - z^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

where we have used the fact that $|z| < r$ and $|z+h| < r$. The expression on the right is the tail end of a convergent series. Therefore, given $\epsilon > 0$ we can find N_1 so that $N > N_1$ implies

$$\left|\frac{E_N(z+h) - E_N(z)}{h}\right| < \frac{\epsilon}{3}.$$

Also since $\lim_{N\to\infty} S_N'(z) = f(z)$, we can find N_2 so that $N > N_2$ implies that

$$|S_N'(z) - f(z)| < \frac{\epsilon}{3}.$$

If we fix N so that both $N > N_1$ and $N > N_2$ hold.
Now $S_N(z)$ is a polynomial and the derivative of a polynomial is obtained by differentiating it term by term. Then we can find $\delta > 0$ so that $|h| < \delta$ implies

$$\left|\frac{S_N(z+h) - S_N(z)}{h} - S_N'(z)\right| < \frac{\epsilon}{3}.$$

Therefore,

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| < \epsilon$$

whenever $|h| < \delta$.

Corollary: The function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is infinitely differentiable on $B(0, R)$ and the higher derivatives are also power series obtained by termwise differentiation and has same radius of convergence R.