MA 201
COMPLEX ANALYSIS
ASSIGNMENT–4&5
AND
PRACTICE PROBLEMS

(1) Let f be an entire function such that $\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = 0$. Show that f is constant.

(2) Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\mathbb{C} \setminus \{0\}$ and bounded on $B(0, \frac{1}{2})$. Show that $\int_{|z|=R} f(z)dz = 0$ for all $R > 0$.

(3) Show that an entire function satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$ is a constant.

(4) Let $g(z)$ be an analytic in $B(0, 2)$. Compute $\displaystyle\int_{|z|=1} f(z)dz$ if

$$f(z) = \frac{a_k}{z^k} + \cdots + \frac{a_1}{z} + a_0 + g(z)$$

where a_i's are complex constants.

(5) Let f be an entire function such that $|f(0)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Then either $f(0) = 0$ or f is constant.

(6) Find the radius of convergence of the following power series:
   (a) $\displaystyle\sum_{n\geq 0} z^{n!}$
   (b) $\displaystyle\sum_{n\geq 0} 2^{n^2} z^n$
   (c) $\displaystyle\sum_{n\geq 0} \frac{(-1)^n}{n} z^{n(n+1)}$
   (d) $\displaystyle\sum_{n\geq 0} a_n z^n$ where $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 3^n & \text{if } n \text{ is even.} \end{cases}$

(7) Find the power series expansion of the following functions about the point $z_0 = 0$ and find its radius of convergence
   (i) $f(z) = \cos^2 z$ (ii) $f(z) = \sinh^2 z$ (iii) $f(z) = \operatorname{Log}(1+z)$
   (iv) $f(z) = \sqrt{z + 2i}$
   (v) $f(z) = \displaystyle\int_0^z \exp(w^2)\, dw$

(8) Find the Taylor series for the function $\frac{1}{z}$ about the point $z_0 = 2$. Then, by differentiating that series term by term, show that $\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n (n+1)\left(\frac{z-2}{2}\right)^n$ for $|z-2| < 2$.

(9) Expand $f(z) = \frac{1}{1-z}$ in a power series about the point $z_0 = 2i$.

(10) If the radius of convergence for the series $\sum_{n=0}^{\infty} a_n z^n$ is R, then find the radius of convergence for the following:
(i) $\sum_{n=0}^{\infty} n^3 a_n z^n$ (ii) $\sum_{n=0}^{\infty} a_n^4 z^n$ (iii) $\sum_{n=0}^{\infty} a_n z^{2n}$ (iv) $\sum_{n=0}^{\infty} a_n z^{7+n}$
(v) $\sum_{n=1}^{\infty} n^{-n} a_n z^n$

(11) Expand each of the following functions about the point $z = 1$ into a power series and find the radius of convergence:
(i) $\frac{z}{z^2-2z+5}$ (ii) $\sin(2z - z^2)$ (iii) $\text{Log}\,(1+z^2)$

(12) Using the Cauchy product of series, find the first four non-zero terms of the Maclaurin series of $e^z/(1-z)$.

(13) Prove or disprove the existence of an analytic function in a neighborhood of the origin satisfying $|f^{(n)}(0)| \geq (n!)^2$, $n = 1, 2, \ldots$.

(14) Suppose f is analytic on the open unit disc D and it satisfies $|f(z)| \leq 1$ for all $z \in D$. Show that $|f'(0)| \leq 1$.

(15) Let z_0 be a zero of order m and n respectively for the analytic functions f and g. Find the nature of the point z_0 for the following functions.
(i) $f+g$ (ii) fg (product) (iii) $\frac{f}{g}$ (iv) $\frac{f}{f'}$ (v) $\frac{f'}{f}$
(vi) $\frac{1}{f}$ (vii) $\exp(1/f(z))$.

(16) Prove or disprove that there exists a non-constant function that is analytic in $|z| < 1$ such that $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for $n = 2, 3, 4, \cdots$.

(17) Prove or disprove that there exists a non-constant function that is analytic in $|z| < 1$ such that $f\left(\frac{1}{n}\right) = f\left(\frac{-1}{n}\right) = \frac{1}{n}$ for $n = 2, 3, 4, \cdots$.

(18) Is there a polynomial $P(z)$ such that $P(z)e^{\frac{1}{z}}$ is an entire function? Justify your answer.

(19) Find the Laurent series of the function $f(z) = \exp\left(z + \frac{1}{z}\right)$ around 0. Hence show that for all $n \geq 0$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta d\theta = \sum_{k=0}^{\infty}\frac{1}{(n+k)!k!}.$$

(20) If f and g are entire functions such that $g\bar{f}$ is entire then either f is constant or $g \equiv 0$.

(21) Find the Laurent series expansion of the following functions about the given points $z = z_0$ or in the given region (specify the region in which the expansion is valid wherever it is necessary).

(a) $z^2\exp(1/z)$ in the neighborhood of $z = 0$

(b) $\dfrac{1}{z^2+1}$ in the neighborhood of $z = -i$

(c) $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ for $0 < |z| < 1$ and for $1 < |z| < 2$.

(22) Let $f(z) = (z+1)^2$ for $z \in \mathbb{C}$. Let R be the closed triangular region with vertices at the points $z = 0$, $z = 2$ and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values.

(23) For the following functions, locate and classify all the singular points.

(i) $\sin\left(\dfrac{1}{z}\right)$ (ii) $\dfrac{1}{\sin\left(\frac{1}{z}\right)}$ (iii) $\cot z - (2/z)$ (iv) $\dfrac{z\exp(1/(z-1))}{\exp(z)-1}$

(24) Using Rouché's theorem prove Fundamental Theorem of Algebra.

(25) Find the isolated singularities and compute the residue of the functions

$$a)\ \frac{e^z}{z^2-1},\quad b)\ \frac{3z}{z^2+iz+2},\quad c)\ \cot\pi z,\quad d)\ \frac{\pi\cot\pi z}{(z+\frac{1}{2})^2}.$$

(26) Find the residues of the function $\dfrac{1}{z^3-z^5}$ at all isolated singular points in $\widehat{\mathbb{C}}$.

(27) Find the residues of $f(z) = \dfrac{e^{imz}}{z^2+a^2}$, ($m$, a real) at its singularities in $\mathbb{C}$.

(28) Show that the residue at the point at infinity for the function $f(z) = \left(\dfrac{z^4}{2z^2-1}\right)\sin\left(\dfrac{1}{z}\right)$ is equal to $(-1/6)$.

(29) Evaluate $\displaystyle\int_C \frac{z\,dz}{\cos z}$ where $C:\ \left|z - \dfrac{\pi}{2}\right| = \dfrac{\pi}{2}$.

(30) Using the Cauchy's residue theorem, evaluate $\displaystyle\int_C \frac{(z^2+3z+2)}{(z^3-z^2)}\,dz$ where $C:\ |z| = 2$.

(31) Using the argument principle, evaluate $\displaystyle\frac{1}{2\pi i}\int_C \cot z\,dz$ where $C: |z| = 7$.

(32) Let $f(z) = (z^3 + 2)/z$. Let $C : z(\theta) = 2e^{i\theta}$, $0 \le \theta \le 2\pi$ be the circle. Let Γ denote the image curve under the mapping $w = f(z)$ as z traverses C once. Determine the change in the argument of $f(z)$ as z describes C once. How many times does Γ wind around the origin in the w-plane and what is the orientation of Γ?

(33) Using Rouche's theorem, find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$ lying in $|z| < 1$.

(34) How many roots of the equation $z^4 - 5z + 1 = 0$ are situated in the domain $|z| < 1$? In the annulus $1 < |z| < 2$?

Practice Problems

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(35) Prove that $\displaystyle\int_0^\infty \frac{dx}{x^4 + a^4} = \frac{\pi}{2a^3\sqrt{2}}, \qquad (a > 0)$

(36) Prove that $\displaystyle\int_0^\infty \frac{x\ \sin(mx)\ dx}{x^2 + a^2} = \frac{\pi}{2}\exp(-ma),\ m > 0$

(37) Prove that $P.V. \displaystyle\int_{-\infty}^{\infty} \frac{x\ dx}{(x^3 + 1)} = \frac{\pi}{\sqrt{3}}$

(38) Prove that $\displaystyle\int_0^\infty \frac{\sin(\pi x)\ dx}{x(1 - x^2)} = \pi$

(39) Using "Indented contour", show that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2 + 1)} = \frac{\pi}{\sqrt{2}}$ by integrating an appropriate branch of the multiple valued function.

(40) Using "key hole contour", show that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2 + 1)} = \frac{\pi}{\sqrt{2}}$ by integrating an appropriate branch of the multiple valued function.

(41) State where the following mappings are conformal.
(i) $w = \sin z$ (ii) $w = z^2 + 2z$.

(42) Show that the mapping $w = \cos z$ is not conformal at $z_0 = 0$.

(43) Find a bilinear transformation which maps $2,\ i,\ -2$ onto $1,\ i,\ -1$.

(44) Find a Mobius transformation which maps $0,\ 1,\ \infty$ onto $i,\ -1,\ -i$.

(45) Find a Mobius transformation which maps $i,\ -1,\ 1$ onto $0,\ 1,\ \infty$.

(46) Find a bilinear transformation which maps $\infty,\ i,\ 0$ onto $0,\ i,\ \infty$.

(47) Show that the transformation $w = \dfrac{z - i}{1 - iz}$ maps the interior of the circle $|z| = 1$ onto the lower halfplane $\mathrm{Im}\,(w) < 0$.

(48) Find the image of the straight line $\mathrm{Re}\,(z) = a$ (constant) in the z-plane under the mapping $w = \dfrac{z - 1}{z + 1}$.