MA 201
COMPLEX ANALYSIS
ASSIGNMENT–2

(1) Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,\ y) = (x,\ -y)$ for all $(x,\ y) \in \mathbb{R}^2$ is differentiable at every point in $\mathbb{R}^2$. View the same function as a complex function. Show that $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}$ for all $z \in \mathbb{C}$ is not differentiable at any point in $\mathbb{C}$.

(2) Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there do not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that
$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$
(Mean value theorem does not extend to complex derivatives).

(3) If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.

(4) Let U be an open set and $f\colon U \to \mathbb{C}$ be a differentiable function. Let $\overline{U} := \{\overline{z}\colon z \in U\}$. Show that the function $g\colon \overline{U} \to \mathbb{C}$ defined by $g(z) := \overline{f(\overline{z})}$ is differentiable on $\overline{U}$.

(5) Derive the Cauchy-Riemann equations in polar coordinates.

(6) Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}$, $f(z) = f(w)$ whenever $|z| = |w|$. Prove that f is a constant function. (Use CR equations in polar coordinates)

(7) Let $f = u + iv$ is an analytic function defined on the whole of $\mathbb{C}$. If $u(x, y) = \phi(x)$ and $v(x, y) = \psi(y)$ prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a \in \mathbb{C}$, $b \in \mathbb{C}$.

(8) Let v be a harmonic conjugate of u. Show that $h = u^2 - v^2$ is a harmonic function.