MA 201
COMPLEX ANALYSIS
ASSIGNMENT–1

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(1) Prove the following:
  (a) Prove that $|z| \leq |\text{Re }(z)| + |\text{Im}(z)| \leq \sqrt{2}\,|z|$.
  (b) $|z_1 + z_2| \leq |z_1| + |z_2|$ and equality holds if and only if one is a nonnegative (real) scalar multiple of the other.
  (c) If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?

(2) Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C}: |z| < 1\}$.

(3) If z and w are in $\mathbb{C}$ such that $\text{Im}(z) > 0$ and $\text{Im}(w) > 0$, show that $|\frac{z-w}{z-\overline{w}}| < 1$.

(4) when does $az + b\overline{z} + c = 0$ has exactly one solution?

(5) If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that $\Pi_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j$.

(6) For each of the following subsets of $\mathbb{C}$, determine whether it is open, closed or neither. Justify your answers.
  (a) $A_1 = \{z \in \mathbb{C} : Re(z) = 1 \text{ and } Im(z) \neq 4\}$
  (b) $A_2 = B(1, 1) \cup B(2, \frac{1}{2}) \cup B(3, \frac{1}{3})$
  (c) $A_3 = \left\{z \in \mathbb{C} : \left|\frac{z-1}{z+1}\right| = 2\right\}$
  (d) $A_4 = \{z \in \mathbb{C} : \sin(Re(z)) < Im(z) < 1\}$

(7) For each of the following subsets of $\mathbb{C}$, determine their interior, exterior and boundary:
  (a) $S_1 = \{z \in \mathbb{C} : |z| < 1 \text{ and } Im(z) \neq 0\} \cup \{z \in \mathbb{C} : |z| > 1 \text{ and } Im(z) = 0\}$
  (b) $S_2 = \left\{r\left(\cos\left(\frac{1}{n}\right) + i \sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{z \in \mathbb{C} : Re(z) < 0\}$

(8) Use $\epsilon - \delta$ definition to prove that

$$\lim_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4(1 + i).$$

Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

(9) Find all points of discontinuity of the following functions:
  (a) $f(z) = \frac{2z-3}{z^2+2z+2}$

(b) $\frac{3z^2+4}{z^4-16}$
(c) $\frac{z}{z^2+1}$

(10) Investigate the convergence of the following sequences:
(a) $a_n = \frac{i^n}{n}$
(b) $a_n = \frac{(1+i)^n}{n}$
(c) $a_n = \frac{n^2 i^n}{n^3+1}$
(d) $a_n = n\left(\frac{1+i}{2}\right)^n$
(e) $a_n = \sqrt{n+2i} - \sqrt{n+i}$