PH101

Lecture -10

31st August 2017

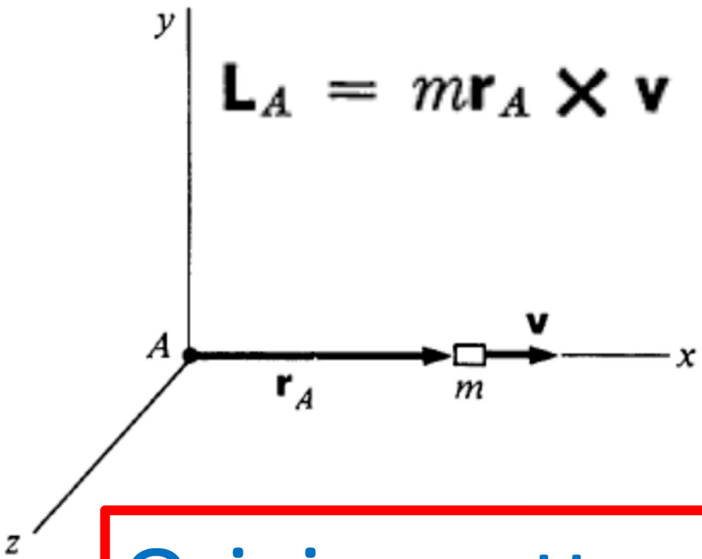
Angular Momentum & Torque

Angular Momentum (Moment of Momentum)

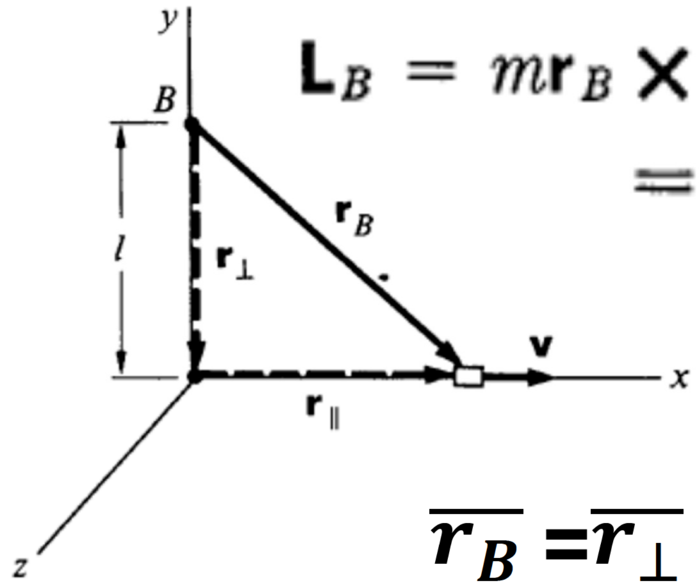
- Angular Momentum of a particle,

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

Direction of ***L*** is perpendicular to the plane of **r** and **p**



$$\mathbf{L}_A = m\mathbf{r}_A \times \mathbf{v} = 0$$

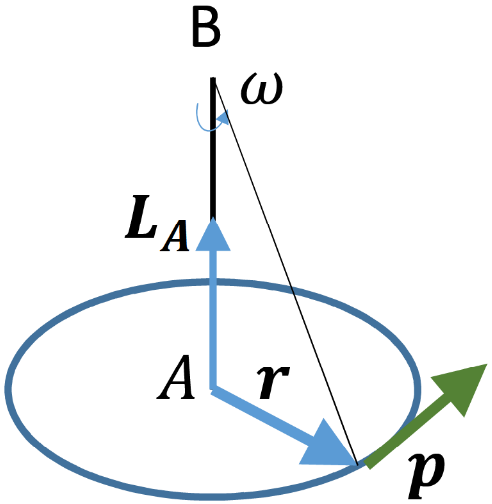


$$\mathbf{L}_B = m\mathbf{r}_B \times \mathbf{v}$$
$$= ml v\hat{\mathbf{k}}.$$

$$\overline{\boldsymbol{r}_B} = \overline{\boldsymbol{r}_\perp} + \overline{\boldsymbol{r}_\parallel}$$

Origin matters!

Angular Momentum of Conical Pendulum



The pendulum is in steady circular motion with constant angular velocity $\omega\hat{\boldsymbol{k}}$.

Angular momentum about A:

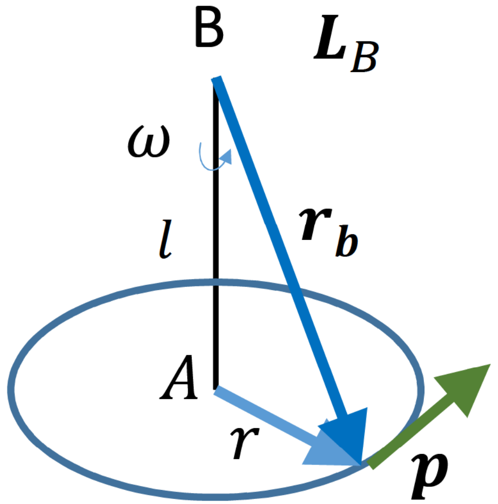
$$\boldsymbol{L_A} = \boldsymbol{r} \times \boldsymbol{p} = rp\hat{\boldsymbol{k}}$$

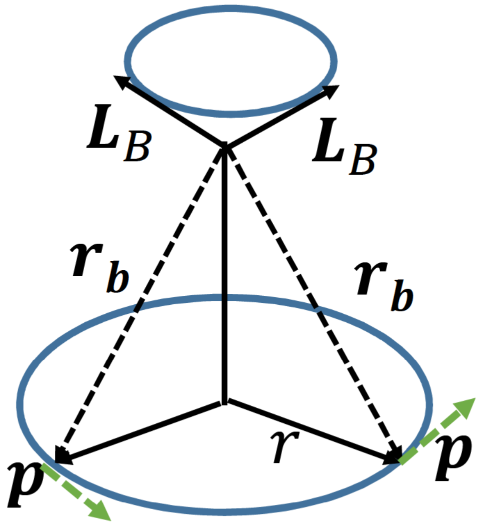
$$p = Mv = Mr\omega$$

$$\boldsymbol{L_A} = Mr^2\omega\ \hat{\boldsymbol{k}}$$

$\boldsymbol{L}_A$ is **constant** in both magnitude and direction.

Angular Momentum of Conical Pendulum





Angular momentum about B:

$$\boldsymbol{L}_{\boldsymbol{B}} = \boldsymbol{r}_{\boldsymbol{b}} \times \boldsymbol{p}$$

Note that $\boldsymbol{r}_{\boldsymbol{b}} = -l\hat{\boldsymbol{k}} + r\hat{\boldsymbol{r}}$ & **p** $= mv\,\hat{\boldsymbol{\theta}}$

$$\boldsymbol{L}_{\boldsymbol{B}} = mv(-l\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\theta}} + r\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}})$$

$$= mv(l\hat{\boldsymbol{r}} + r\hat{\boldsymbol{k}}) = mr\omega(l\hat{\boldsymbol{r}} + r\hat{\boldsymbol{k}})$$

$\boldsymbol{L}_B$ is constant in magnitude **but direction** is changing because $\hat{\boldsymbol{r}}$ is changing!

The z-component of $\boldsymbol{L}_B$ $(mr^2\omega)$ is constant but the horizontal component $(mlr\omega)$ changes its **direction**!

Torque

For a force, F acting on a particle at r, the torque is

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

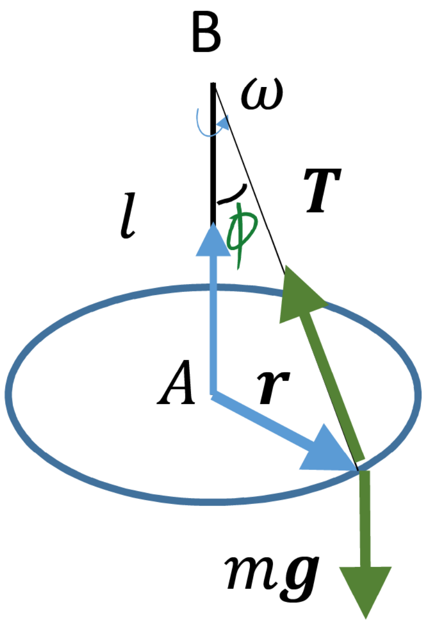
Like angular momentum, the **torque depends on the choice of origin.**

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p})$$

$$= \cancel{\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p}} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt}$$

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F}$$

Torque on a Conical Pendulum



The pendulum is in steady circular motion with constant angular velocity $\omega\hat{\boldsymbol{k}}$.

Angular momentum about A:

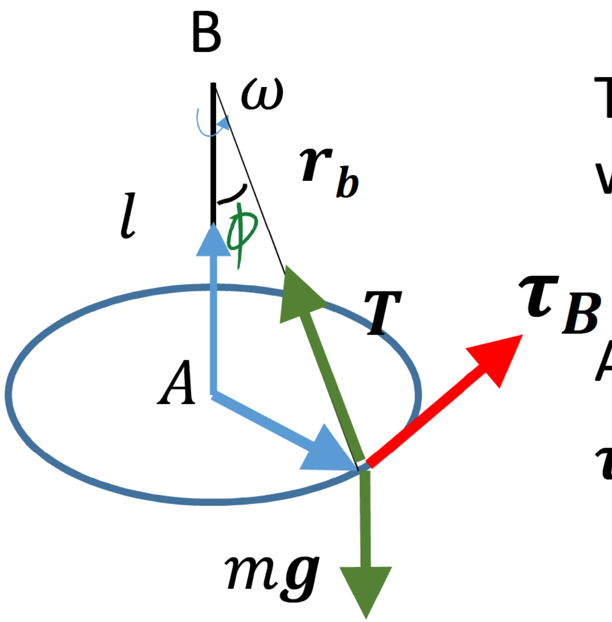
$$\boldsymbol{\tau}_A = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times (\boldsymbol{T} + mg(-\hat{\boldsymbol{k}}))$$

T Cosϕ **–**mg **= 0 (vertical, $\hat{\boldsymbol{k}}$)**

T Sinϕ (horizontal; $-\hat{\boldsymbol{r}}$)

$\boldsymbol{\tau}_A = 0$ (**L**$_{\mathbf{A}}$ is constant!)

Torque on a Conical Pendulum



The pendulum is in steady circular motion with constant angular velocity $\omega\hat{\boldsymbol{k}}$.

Angular momentum about A:

$$\boldsymbol{\tau}_B = \boldsymbol{r}_b \times \boldsymbol{F} = \boldsymbol{r}_b \times \boldsymbol{T} + \boldsymbol{r}_b \times mg(-\hat{\boldsymbol{k}})$$

$$\boldsymbol{r}_b = -l\hat{\boldsymbol{k}} + r\hat{\boldsymbol{r}}$$

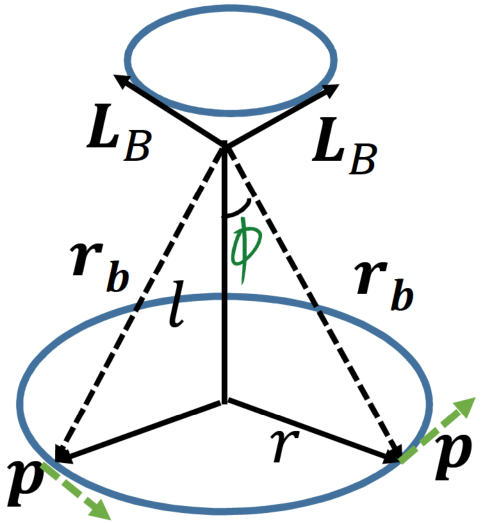
$$\boxed{\hat{\boldsymbol{r}}, \hat{\theta}, \hat{\boldsymbol{k}}}$$

$$\boldsymbol{\tau}_B = -l\hat{\boldsymbol{k}} \times mg(-\hat{\boldsymbol{k}}) + r\hat{\boldsymbol{r}} \times mg(-\hat{\boldsymbol{k}})$$

$$\boldsymbol{\tau}_B = \mathrm{mgr}\ \hat{\theta} \quad = \frac{d\boldsymbol{L}_B}{dt}$$

($\boldsymbol{\tau}_B$ changes its direction!)

Torque of Conical Pendulum



Angular momentum about B:

$$\boldsymbol{L_B} = mr\omega(l\hat{\boldsymbol{r}} + r\widehat{\boldsymbol{k}})$$

$$\frac{d\boldsymbol{L_B}}{dt} = mr\omega l \frac{d\hat{\mathbf{r}}}{dt} = mrl\omega\omega\hat{\theta}$$

T Cosϕ =mg **(vertical,** $\widehat{\boldsymbol{k}}$**)**

T Sinϕ= $mr\omega^2$ (horizontal; $-\hat{\boldsymbol{r}}$)

g/$r\omega^2$ = cot ϕ = l/r

ω^2 = g/l

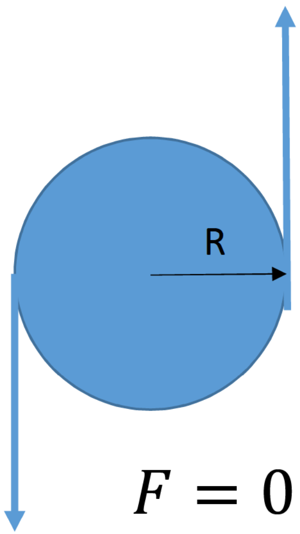
$$\frac{d\boldsymbol{L_B}}{dt} = mrl\omega^2\hat{\theta} = mgr\hat{\theta} = \boldsymbol{\tau_B}$$

Force & Torque

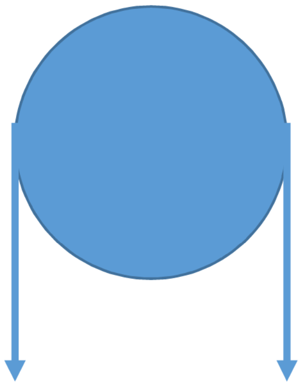
Torque and force are very different quantities (KK)

There can be a torque on a system with zero net force

And there can be net force on a system with zero net torque.



$$F = 0$$
$$\tau = 2Rf$$



$$F = 2f$$
$$\tau = 0$$

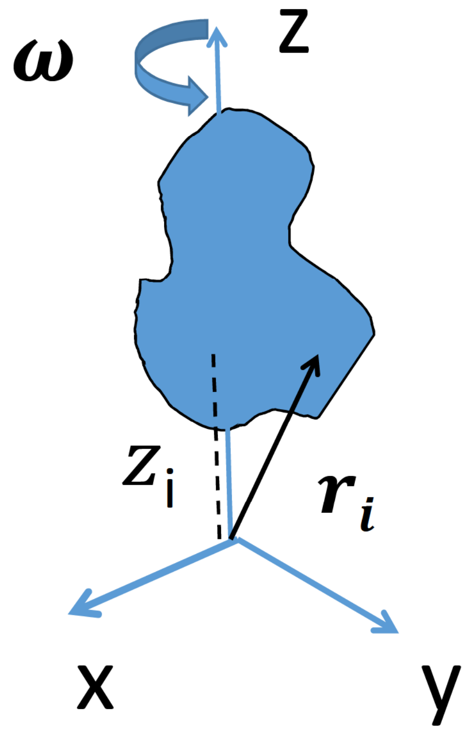
Rigid Bodies

- Many particle system
- Interparticle distance for all pairs of particles $|r_{ij}| = c_{ij}$
- Retains shape and size
- 6 degrees of freedom

FIXED AXIS ROTATION

- All particles rotate about an axis fixed in space
- All particles have the same angular speed ω
- Let us choose the z-axis as the axis of rotation, with $\boldsymbol{\omega} = \omega\hat{k}$

Rotation about a Fixed Axis



$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i$$

$$\boldsymbol{L} = \sum_i m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) = \sum_i m_i r_i^2 \boldsymbol{\omega} - \sum_{\boldsymbol{i}} m_i (\boldsymbol{r}_i \cdot \boldsymbol{\omega}) \boldsymbol{r}_i$$

$$\boldsymbol{L}_z = \sum_i m_i \left(r_i^2 - z_i^2\right) \omega \hat{k} = \sum_i m_i \left(x_i^2 + y_i^2\right) \omega \hat{\boldsymbol{k}}$$

$$\boldsymbol{L}_z = I_{zz} \omega \hat{\boldsymbol{k}}$$

Moment of Inertia

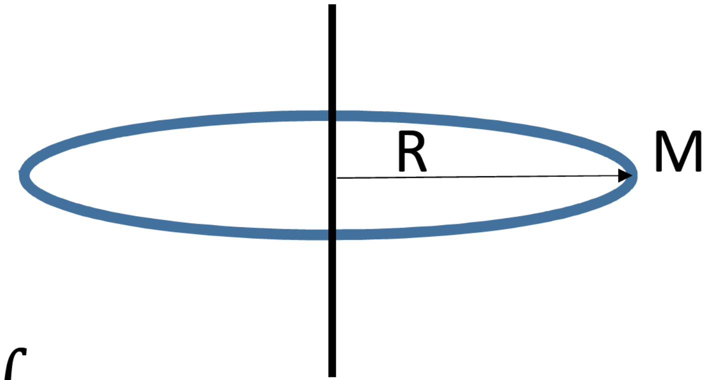
- I_{zz} is called the moment of inertia about the z-axis
- $\sum_i m_i\left(x_i^2 + y_i^2\right) = I_{zz}$ is a purely geometric quantity. Does not depend on the motion.
- For continuous bodies

$$I_{ZZ} = \int dm(x^2 + y^2)$$

$$I_{ZZ} = \int \rho(x^2 + y^2)dV$$

Where $\rho(\boldsymbol{r})$ is the density and dV is the volume element.

Example1



A hoop/ring of mass M and radius R.

$$I_{ZZ} = \int dm(x^2 + y^2) = \int dm\, R^2 = MR^2$$

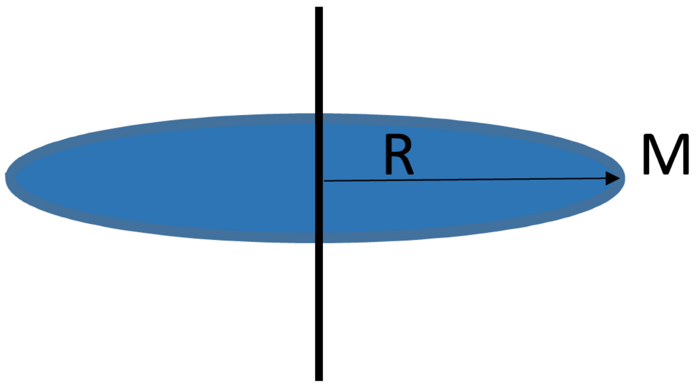
Or, $I_{ZZ} = \int(\lambda R d\theta)\ R^2 = R^3 2\pi\, \lambda = MR^2$ $\lambda = M/2\pi R$

Example2

A thin disk of mass M and radius R.

$$I_{ZZ} = \int \sigma \, dx \, dy \, (x^2 + y^2) = \int r^2 \, \sigma \, dr \, r d\theta = \frac{R^4}{4} 2\pi \, \sigma = \frac{MR^2}{2}$$

$$\sigma = \frac{M}{\pi R^2}$$



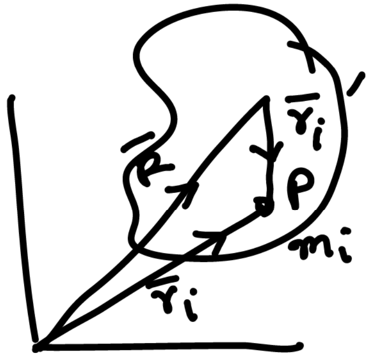
System of Particles (General Motion)

$$\bar{L} = \sum_{i=1} \bar{r}_i \times m_i \dot{\bar{r}}_i$$

$$\bar{R} = \frac{\sum m_i \bar{r}_i}{M} \quad (CM)$$

$$\bar{r}_i = \bar{R} + \bar{r}_i'$$

$$\dot{\bar{r}}_i = \dot{\bar{R}} + \dot{\bar{r}}_i'$$



$$\bar{L} = \sum m_i (\bar{R} + \bar{r}_i') \times (\dot{\bar{R}} + \dot{\bar{r}}_i')$$

$$= \sum m_i (\bar{R} \times \dot{\bar{R}}) + \sum m_i \bar{R} \times \dot{\bar{r}}_i' + \sum m_i \bar{r}_i' \times \bar{R} + \sum m_i \bar{r}_i' \times \dot{\bar{r}}_i'$$

$$= \underbrace{\bar{R} \times M \dot{\bar{R}}}_{\bar{L}_0} + \bar{R} \times \sum m_i \dot{\bar{r}}_i' + \left(\sum m_i \bar{r}_i'\right) \times \bar{R} + \underbrace{\sum \bar{r}_i' \times m_i \dot{\bar{r}}_i'}_{\bar{L}_{CM}}$$

But

$$M\bar{R} = \sum m_i \bar{r}_i = \sum m_i(\bar{R} + \bar{r}_i{}') = M\bar{R} + \sum m_i \bar{r}_i{}'$$

ie $\underline{\underline{\sum_i m_i \bar{r}_i{}' = 0}}$

So $$\boxed{\bar{L} = \bar{R} \times M\bar{V}_{cm} + \sum_i \bar{r}_i{}' \times \bar{p}_i{}'}$$

$\bar{p}_i{}' = m_i \dot{\bar{r}}_i{}' = m_i \bar{v}_i{}'$
w.r.to CM

Or $$\underline{\underline{\bar{L} = \bar{L}_0 + \bar{L}_{cm}}}$$

Note: $\bar{L}_0$ is the $\bar{L}$ of the Centre of Mass.

ie, $\bar{L}_0 = \bar{R} \times M\bar{V}_{cm} = \bar{R} \times \bar{P}_{cm}$

$\bar{L}_{cm}$ is the $\sum \bar{r}_i{}' \times \bar{p}_i{}'$ <u>with respect to the</u> <u>CM</u>!

__Torque__

$$\bar{\tau} = \sum \bar{r}_i \times \bar{f}_i$$

$\bar{f}_i = \bar{f}_i^{ext} + \bar{f}_i^{int}$ on i^{th} particle

$$= \sum (\bar{R} + \bar{r}_i') \times \bar{f}_i$$

$$= \bar{R} \times \sum_i \bar{f}_i + \sum \bar{r}_i' \times \bar{f}_i$$

$$= \bar{R} \times \left(\sum_i f_i^{int} + \sum_i f_i^{ext}\right) + \sum \bar{r}_i' \times \bar{f}_i$$

But $\sum_i \bar{f}_i^{int} = 0$!

$$\bar{\tau} = \bar{R} \times \bar{F} + \sum \bar{r}_i' \times \bar{f}_i^{int} + \sum \bar{r}_i' \times f_i^{ext}$$

$\bar{F}$ - is the __total__ __external__ force on the body!

$\sum \bar{r}_i' \times \bar{f}_i^{int}$ | $\bar{f}_i^{int} = \sum_{j \neq i} \bar{f}_{ij}$ (force on i due to j)

has terms like $\bar{r}_i' \times \bar{f}_{ij} + \bar{r}_j' \times \bar{f}_{ji} = (\bar{r}_i' - \bar{r}_j') \times \bar{f}_{ij}$

$\bar{f}_{ji} = -\bar{f}_{ij}$

Now if the internal forces between particles i & j act along the line connecting them $(\bar{r}_i' - \bar{r}_j')$

then $(\bar{r}_i' - \bar{r}_j') \times \bar{F}_{ij} = 0$

So contribution to $\bar{\tau}$ due to internal forces vanish!

So $$\boxed{\bar{\tau} = \bar{R} \times \bar{F} + \sum \bar{r}_i' \times \bar{f}_i^{ext}}$$