## PH101

## Lecture 7

More examples on Lagrange's equation; generalized momentum, Cyclic coordinates
Conservation of Momentum.
I. (a) Recognize, \& obtain the constraint relations, (b) determine th
 DOF, and (c) choose appropriate generalized coordinates!
II. Write down the total kinetic energy $\boldsymbol{T}$ and potential energy $\boldsymbol{V}$ of the whole system in terms of the Cartesian coordinates, to begin with!

$$
\begin{aligned}
& \quad T=\sum_{i=1}^{N} \frac{1}{2} m_{i}\left(\dot{x}_{i}^{2}+\dot{y}_{i}^{2}+z_{i}^{2}\right) \quad \& \quad V= \\
& \text { III. Obtain appropriate transformation equations }
\end{aligned}
$$

(Cartesian $\rightarrow$ generalized coordinates) using constraint relations: $z_{i}=z_{i}\left(q_{1}, \ldots, q_{n}, t\right)$
IV. Convert $T$ and $V$ from Cartesian to suitable generalized -coordinates $\left(\boldsymbol{q}_{\boldsymbol{j}}\right)$ and generalized velocities $\left(\dot{q}_{j}\right)$ to write L as,

$$
\boldsymbol{L}\left(\boldsymbol{q}_{\boldsymbol{j}}, \dot{\boldsymbol{q}}_{\boldsymbol{j}}, \boldsymbol{t}\right)=\boldsymbol{T}\left(\boldsymbol{q}_{\boldsymbol{j}}, \dot{\boldsymbol{q}}_{\boldsymbol{j}}, \boldsymbol{t}\right)-\mathrm{V}\left(\boldsymbol{q}_{\boldsymbol{j}}\right) \quad j=1, \mathrm{n}
$$

V. Now Apply E-L equations:

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{i}}\right)-\frac{\partial L}{\partial q_{i}}=0 \quad \text { for each } \mathrm{j}=1, \mathrm{n}!
$$

Before going further let's see the Lagrange's equations recover Newton's $\mathbf{2}^{\text {nd }}$ Law, if there are NO constraints!
Let a particle of mass, $m$, in 3-D motion under a potential, $V(x, y, z)$
If No constraints, then its, $\mathrm{DOF}=3$; Generalized coordinates: $(x, y, z)$
Now,

$$
L=\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)-V(x, y, z)
$$

Corresponding E-L equations are,

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0
$$

$$
\left(\frac{\partial L}{\partial \dot{x}}\right)=m \dot{x}=p_{x} \quad \text { - the } \mathrm{x} \text {-component of linear momentum! }
$$

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{d p_{x}}{d t} \quad \& \quad \frac{\partial L}{\partial x}=-\frac{\partial V}{\partial x}=\boldsymbol{F} \boldsymbol{X} \text {-the x-component of force! }
$$

$$
F_{x}=\frac{d p_{x}}{d t}
$$

## Lagrange's equation: Example 3



Initial conditions!
At time $t=0$ : the wedge is stationary and at a distance $l$ from the origin, and the mass $m$ is gently placed at the top point of the Wedge!

## Lagrange's equation: Example 3



$$
\begin{aligned}
& \begin{array}{l}
\text { Four constrains: } z_{M}=0 ; y_{M}=0 ; z_{m}=0 ; \\
\frac{h-y_{m}}{x_{m}-x_{M}}=\tan \alpha=\text { constant }
\end{array} \\
& \qquad \begin{array}{l}
x_{m}=x_{M}+q \cos \alpha ; \\
y_{m}=h-q \sin \alpha
\end{array}
\end{aligned}
$$

Step-1: Find the degrees of freedom and choose suitable generalized coordinates
One particle $N=2$, no. of constrains $(k)=4$;
So $D O F=3 \times 2-4=2$.
The distance of the wedge from origin ( $x_{M}$ ) and distance slipped by the block $(q)$ can serve as generalized coordinates of the system.

Only translation of the given rigid bodies are considered, thus for the calculation of degrees of freedom both of them are considered as point particles.

## Lagrange's equation: Example 3

Step-2: Find out transformation relations

$$
T=\frac{1}{2} m\left(\dot{x}_{m}{ }^{2}+\dot{y}_{m}{ }^{2}\right)+\frac{1}{2} M \dot{x}_{M}{ }^{2} ; \quad V=m g y_{m}
$$

Step-3: Write $T$ and $U$ in Cartesian

$$
\begin{array}{ll}
x_{m}=x_{M}+q \cos \alpha ; & y_{m}=h-q \sin \alpha \\
\dot{x}_{m}=\dot{x}_{M}+\dot{q} \cos \alpha ; & \dot{y}_{m}=-\dot{q} \sin \alpha
\end{array}
$$

Step-4: Convert $T$ and $V$ in generalized coordinate using transformations

$$
\begin{aligned}
& T=\frac{1}{2} m\left[\dot{x}_{M}{ }^{2}+\dot{q}^{2}+2 \dot{x}_{M} \dot{q} \cos \alpha\right]+\frac{1}{2} M \dot{x}_{M}{ }^{2} ; \\
& \mathrm{V}=m g(h-q \sin \alpha)
\end{aligned}
$$

## Lagrange's equation: Example 3

Step-5: Write down Lagrangian

$$
\begin{gathered}
L=T-V \\
L=\frac{1}{2} m\left[\dot{x}_{M}{ }^{2}+\dot{q}^{2}+2 \dot{x}_{M} \dot{q} \cos \alpha\right]+\frac{1}{2} M \dot{x}_{M}{ }^{2}-m g(h-q \sin \alpha)
\end{gathered}
$$

Step-5: Write down Lagrange's equation for each generalized coordinates

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}_{M}}\right)-\frac{\partial L}{\partial x_{M}}=0 ; \quad \frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0
$$

From eqn 1

$$
\begin{align*}
& \frac{d}{d t}\left[m \dot{x}_{M}+m \dot{q} \cos \alpha+M \dot{x}_{M}\right]=0  \tag{1}\\
& (m+M) \ddot{x}_{M}+m \ddot{q} \cos \alpha=0 \tag{2}
\end{align*}
$$

From eqn 2

$$
\begin{align*}
& \frac{d}{d t}\left[m \dot{q}+m \dot{x}_{M} \cos \alpha\right]-[m g \sin \alpha]=0 \\
& m\left(\ddot{q}+\ddot{x}_{M} \cos \alpha\right)-m g \sin \alpha=0 \tag{3}
\end{align*}
$$

## An Interesting point: Example 3

$$
\frac{\partial L}{\partial \dot{x}_{M}}=m \dot{x}_{M}+m \dot{q} \cos \alpha+M \dot{x}_{M}=\text { constant }!
$$

But what's this quantity?
The total linear momentum, say $\boldsymbol{P}_{x}$ !
So Lagrange's equation tells us that the total linear momentum is conserved! We didn't have to impose it to solve!

From the Initial conditions given: $x_{M}=l \dot{x}_{M}=0 ; q=0 ; \dot{q}=0$ Initial $P x=0$; So it any other time later!

$$
\dot{x}_{M}=\frac{-m \dot{q} \cos \alpha}{(M+m)}=>\ddot{x}_{M}=\frac{-m \ddot{q} \cos \alpha}{(M+m)}
$$

This shall be substituted in eq (3): $\quad\left(\ddot{\boldsymbol{q}}+\ddot{\boldsymbol{x}}_{\boldsymbol{M}} \cos \alpha\right)=\boldsymbol{g} \sin \alpha$

## And, Solve the problem completely! (It's left to you to verify with the Newtonian Scheme!)

In some cases further time derivative (such as equation (2)) may not be unnecessary!

## Generalized momentum: A few points

Generalized velocity is the rate of charge of generalized coordinate $\dot{q}_{j}=\frac{d q_{j}}{d t}$

## Generalized momentum is not the mass multiplied by generalized velocity.



Unit/dimension of the generalized momentum depends on generalized coordinate under consideration.

Generalized definition of momentum allows to consider non-mechanical systems, for example EM field. Example: charged particle in EM field $\vec{p}=m \vec{v}+e \vec{A}$

## Generalized momentum

Lagrangian of a free particle

$$
L=\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)
$$

Thus $\frac{\partial L}{\partial \dot{x}}=m \dot{x}$, now $m \dot{x} \rightarrow x$ component of linear momentum $\left(p_{x}\right)$
$p_{x}=m \dot{x}=\frac{\partial L}{\partial \dot{x}} ;$ Similarly, $p_{y}=\frac{\partial L}{\partial \dot{y}}$ and $p_{z}=\frac{\partial L}{\partial \dot{z}}$
Lagrangian of a freely rotating wheel with moment of inertia $I$ is

$$
L=\frac{1}{2} I \dot{\theta}^{2}
$$

And $\frac{\partial L}{\partial \dot{\theta}}=I \dot{\theta} \rightarrow$ Angular momentum
In both the examples, momentum was the derivative of the Lagrangian with respect to generalized velocity.

Generalized momentum associated with generalized coordinate $q_{j}$ by

$$
p_{j}=\frac{\partial L}{\partial \dot{q}_{j}} \quad \longrightarrow \begin{aligned}
& \text { Also known as conjugate momentum } \\
& \text { or canonical momentum }
\end{aligned}
$$

## Cyclic coordinates

If a particular coordinate does not appear in the Lagrangian, it is called 'Cyclic' or 'Ignorable' coordinate.

Example 1: Lagrangian of a point mass under gravity,

$$
L=\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)-m g z
$$

Since neither $x$ nor $y$ appear in the Lagrangian, they are cyclic. Hence $P_{x} \& P_{y}$ will be conserved!

Example 2: Lagrangian for a planet of mass $m$ orbiting around the sum (mass M):

$$
L=\frac{1}{2} m\left(\dot{r}^{2}+r^{2} \dot{\theta}^{2}\right)+\frac{G M m}{r}
$$



Since $\theta$ does not appear in the Lagrangian, it is cyclic coordinate. Hence $P_{\theta}=\frac{\partial L}{\partial \dot{\theta}_{j}}=m r^{2} \dot{\theta} \quad$ ( $\equiv$ Ang. Momentum! -will be conserved!)

## Cyclic coordinates and conservation of conjugate momentum

- If there is no explicit dependence of $L$ on generalized coordinate $q_{j}$, then

$$
\frac{\partial L}{\partial q_{j}}=0
$$

Thus Lagrange's equation corresponding to cyclic coordinate become,

$$
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{q}_{j}}\right)=0 \Rightarrow \frac{d p_{i}}{d t}=0
$$

Hence, $\boldsymbol{p}_{\boldsymbol{j}}=\mathbf{c o n s t a n t}$

## Lagrange's equation: Example 4



A bead is free to slide along a frictionless hoop of radius $R$. The hoop rotates with constant angular speed $\omega$ around a vertical diameter. Find the equation of motion for the position of the bead.

## Lagrange's equation: Example 4



Hence number of generalized coordinates must be one.
Choice of Generalized coordinate: ' $\theta$ ', which the angle of particle with rotation axis (z-axis) of hoop.

## Lagrange's equation: Example 4

Step-2: Find out transformation relations

$$
x=R \sin \theta \cos \omega t ; y=R \sin \theta \sin \omega t ; z=R \cos \theta
$$

$$
\begin{gathered}
\dot{x}=R \cos \theta \cos \omega t \dot{\theta}-R \omega \sin \theta \sin \omega t \\
\dot{y}=R \cos \theta \sin \omega t \dot{\theta}+R \omega \sin \theta \cos \omega t \\
\dot{z}=-R \sin \theta \dot{\theta}
\end{gathered}
$$

Step-3: Write $T$ and $V$ in Cartesian

$$
T=\frac{1}{2} m\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right) ; \quad \& \quad V=m g z
$$

Step-4:Convert $T$ and $V$ to generalized coordinate, either using, (a)transformation at Step\#2 Or,
(b) in this case employing spherical polar equations.

$$
\begin{aligned}
& T=\frac{1}{2} m\left[R^{2} \dot{\theta}^{2}+R^{2} \omega^{2} \sin ^{2} \theta\right] \\
& \mathrm{V}=m g R \cos \theta
\end{aligned}
$$

## Example 4: continue

Step-5: Write down Lagrangian

$$
\begin{gathered}
L=T-V \\
L=\frac{1}{2} m\left[R^{2} \dot{\theta}^{2}+R^{2} \omega^{2} \sin ^{2} \theta\right]-m g R \cos \theta
\end{gathered}
$$

Step-5: Lagrange's equation

$$
\begin{gathered}
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \\
\frac{\partial L}{\partial \dot{\theta}}=m R^{2} \dot{\theta} \quad \& \frac{\partial L}{\partial \theta}=m R^{2} \omega^{2} \sin \theta \cos \theta+m g R \sin \theta \\
\frac{d}{d t}\left[m R^{2} \dot{\theta}\right]-\left[m R^{2} \omega^{2} \sin \theta \cos \theta+m g R \sin \theta\right]=0 \\
m R^{2} \ddot{\theta}-\left[m R^{2} \omega^{2} \sin \theta \cos \theta+m g R \sin \theta\right]=0
\end{gathered}
$$

## Example-5

A mass $M$ slides down a frictionless plane inclined at angle $\alpha$. A pendulum, with length $l$, and mass $m$, is attached to $M$. Find the equations of motion.


## Questions?

