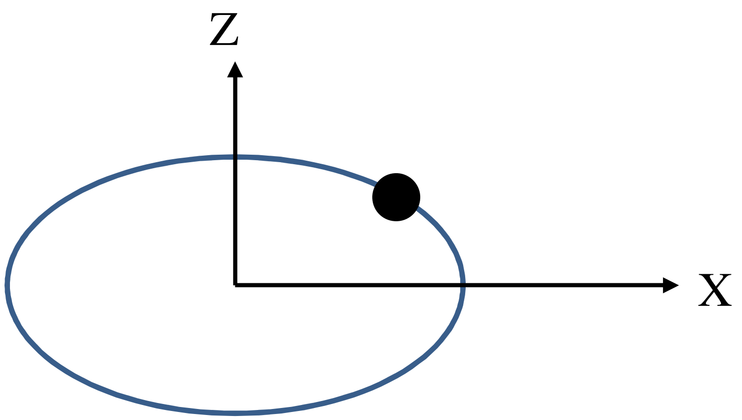
PH101: PHYSICS1

Lecture 5

Constrains, Degree's of freedom and generalized coordinates

Constrains

Motion of particle not always remains free but often is subjected to given conditions.



A particle is bound to move along the circumference of an ellipse in XZ plane.

At all position of the particle, it is bound to obey the condition $\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$

Constrains: Condition or restrictions imposed on motion of particle/particles

Classification of constrains

- **Holonomic Constrains:** Expressible in terms of equation involving coordinates and time (may or may not present),

I,e. $\boldsymbol{f(q_1, \ldots .q_n, t) = 0}$; where q_i are the instantaneous coordinates

- **Non-holonomic constrains :** Constrains which are not holonomic

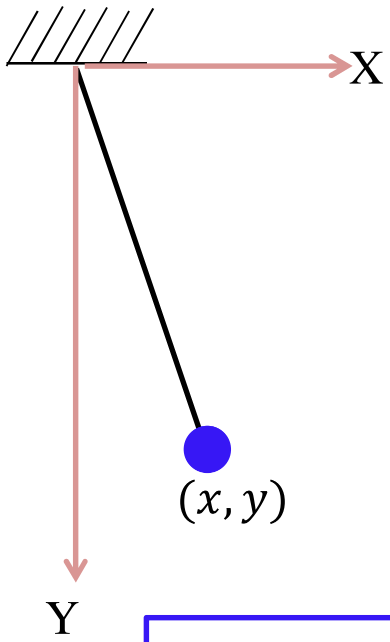
Two types of constrains are there in this category

(i) Equations involving velocities: $\boldsymbol{f(q_1, \ldots, \dot{q}_1, \ldots ., \dot{q}_n, t) = 0}$,
(& those cannot be **reduced** to the holonomic form!).

(ii) Constraints as *in-equalities*,
An example, $\boldsymbol{f(q_1, \ldots ., q_n, t) < 0}$

In both type of constrains (holonomic/non-holonomic) time may or may not be present explicitly.

Pendulum



- *Constrain equations*

$$x^2 + y^2 = l^2$$
$$x = \sqrt{l^2 - y^2}$$

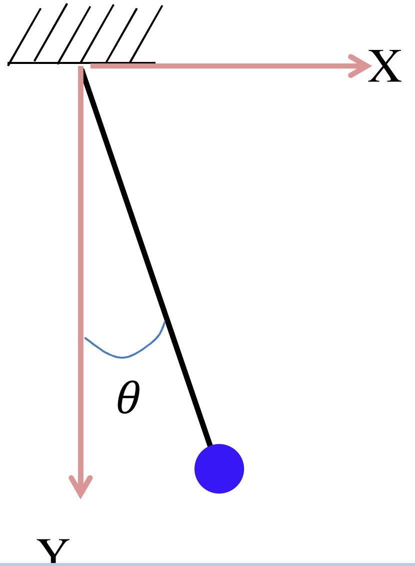
- One can not change x **independently**, any change in x will automatically change y.

$\boldsymbol{x, y}$ ***are not independent*** **due to presence of constrains**

Independent coordinates: If you fix all but one coordinate and still have a continuous range of movement in the free coordinate.

If you fix y_1, leaving x_1 free, then there is no continuous range of x_1 possible. In fact in this case there will not be any motion if you fix y_1

Degree of Freedom &Generalized coordinate



- If you choose θ as the only coordinate, it can represent entire motion of the bob in XY plane

- In this problem, only one coordinate θ is sufficient which is sole independent coordinate.

Degree of Freedom (DOF): no of independent coordinate required to represent the entire motion = $3 \times (no\ of\ particles) - no.of\ constrains$ =3-2=1

In this case no. of particle=1
No. of constrains =2 $[x^2 + y^2 = l^2$ and $z = 0]$

DOF =1; Generalized Coordinate= θ

Degree's of freedom

- **Degree's of freedom (DOF)**: No. of independent coordinates required to completely specify the dynamics of particles/system of particles is known as degree's of freedom.

- Degree's of freedom $=$ $3 \times (no.of\ particles) - (No.of\ holonomic\ constrains)$

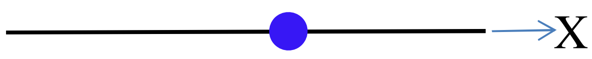
$$= 3N - k$$

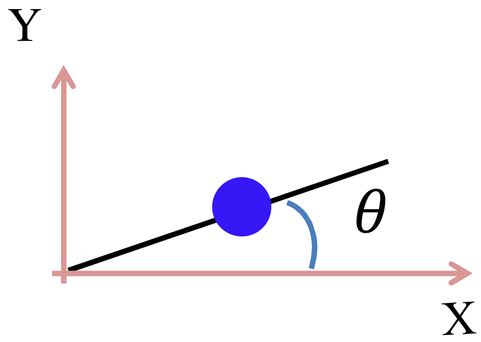
Where

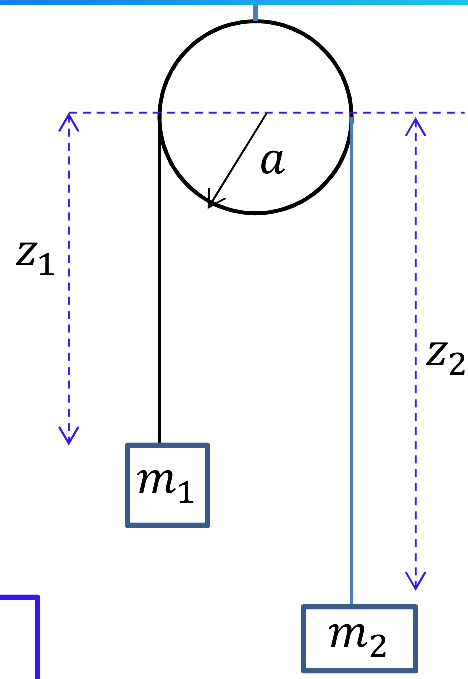
N= No. of particles

k =No. of constrains.

Holonomic constrains







Particle moving along a line (*say X-axis*)

Constrain equations
$$y = 0; z = 0$$

DOF =1;
GC= x

A particle is moving along a straight wire, making an angle With x-axis.

Constrain equations
$$y = x\,tan(\theta);$$
$$z = 0$$

DOF =1; GC = x or y

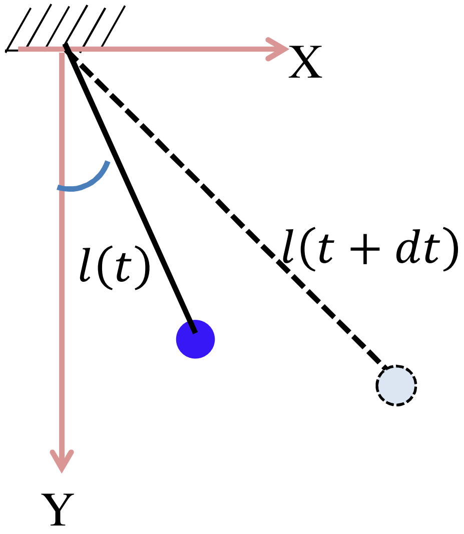
Atwood's machine

Constrain equations
$$z_1 + z_2 + \pi a = l$$
$$x_1 = 0;\ y_1 = 0$$
$$x_2 = 0;\ y_2 = 0$$

DOF =1;
GC = z_1 or z_2

General form of these constrain equations, $f(q_1, \ldots, q_n) = 0$

Pendulum of varying length!



The length of the string is changing with time $l(t)$ and **is known**.

General form of these constrain equations $f(q_1, \dots, q_n, t) = 0$

Pendulum with stretchable string, the bob is constrain to move in a plane

$Constrain\ equations$

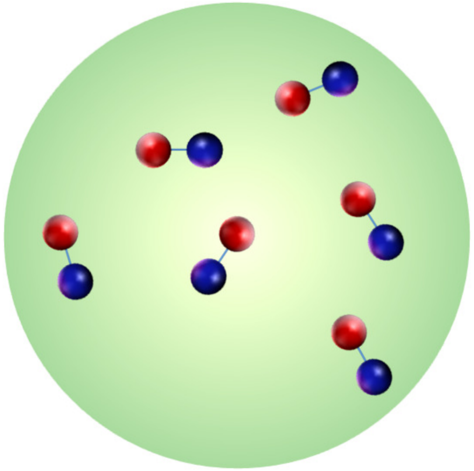
$$x^2 + y^2 = l^2(t)$$

$$z = 0$$

DOF =1; GC = θ

Non-holonomic constraint

Gas molecules confined within a spherical container of radius R

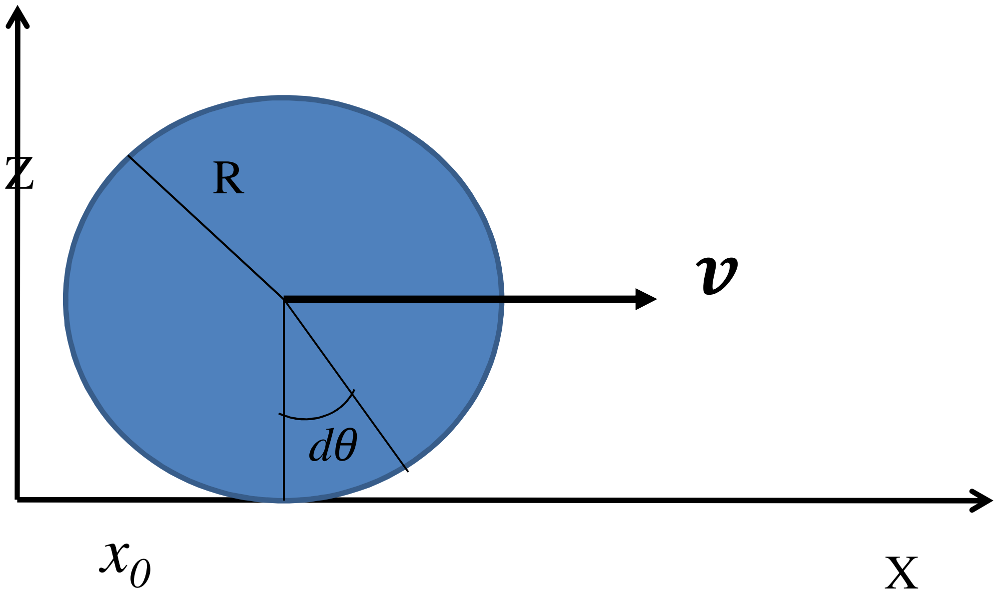


Constrain condition $r_i \leq R$

Inequality!

Rolling Constraint

Rolling of a disc without slipping



$$\boldsymbol{v} = \boldsymbol{R}\dot{\theta}$$

$$dx = Rd\theta$$

x - $R\theta = x_0$ (constraint relation)

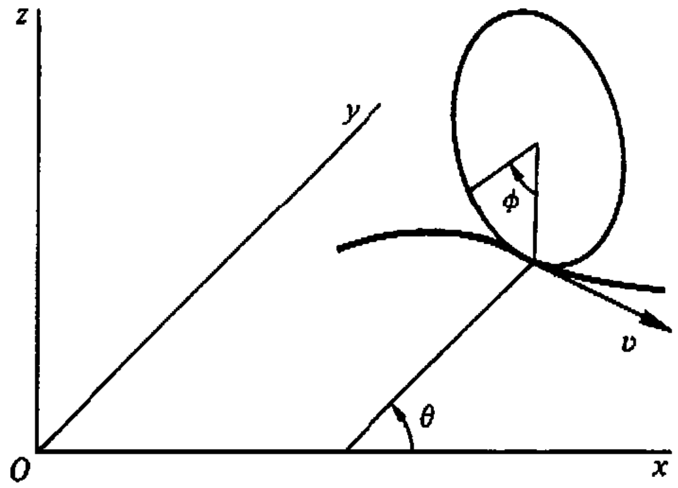
DOF =1; GC = θ

Other *Constrains*:
$\boldsymbol{y = 0}$; $\boldsymbol{z = R}$; $\varphi = \boldsymbol{0}$; $\psi = \boldsymbol{0}$;

More complicated constraint

Speed,

$$v = R\dot{\varphi}$$

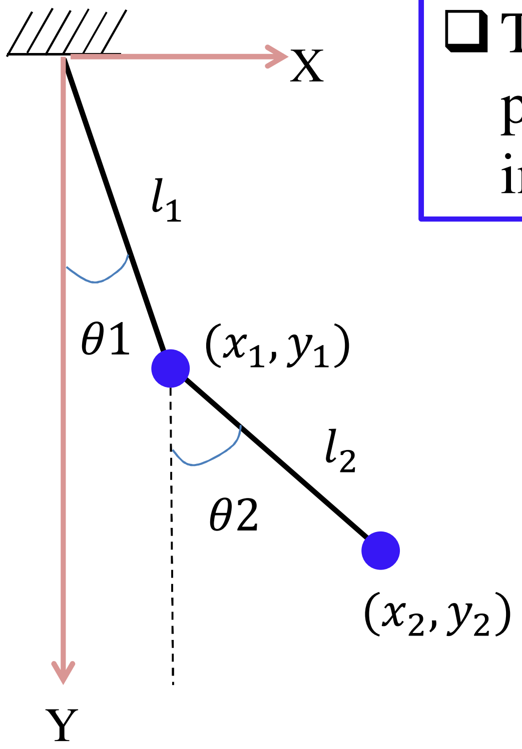


$$\dot{x} = v \ \sin\theta = R\dot{\varphi}\sin\theta$$

$$\dot{y} = -v \ \cos\theta = -R\dot{\varphi}\cos\theta$$

Velocity dependence that can't be integrated out!
Non-holonomic!

Double pendulum



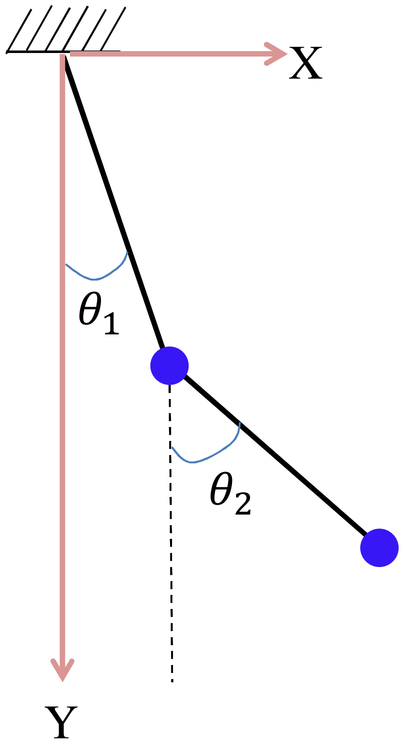
- To describe the motion double pendulum in XY plane, one needs four coordinates (x_1, y_1, x_2, y_2) in Cartesian coordinate system.

- The Cartesian coordinates are **not independent**, they are related by constrain equations

$$x_1{}^2 + y_1{}^2 = l_1{}^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2{}^2$$

If you fix y_1, x_2, y_2 leaving x_1 free, then there is no continuous range of x_1 possible. In fact in this case there will not be any motion by fixing three coordinates leaving one as free.

Generalizer coordinates



- If you choose $\theta_1$and θ_2 as the coordinates, then they can adequately describe the motion of double pendulum at any instant. (they are complete)

No. of constrains = 4
$z_1 = 0;\ z_2 = 0;$
${x_1}^2 + {y_1}^2 = {l_1}^2;$
$(x_2 - x_1)^2 + (y_2 - y_1)^2 = {l_2}^2$

DOF: No. of independent coordinates required to completely specify the motion
$= 3 \times (no.of\ particles) - (No.of\ constrains)$
$= 3 \times 2 - 4 = 2$

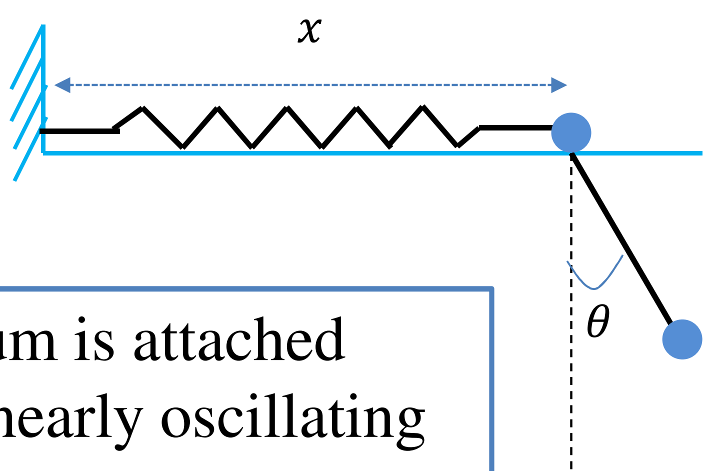
Generalizer coordinates:$\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$

Generalized coordinate?

❑ **Generalized coordinate**

- non necessarily a distance
- Not necessarily an angle.
- **Not necessarily belong to a particular coordinate system!**
  (Cartesian, Cylindrical, Polar or Spherical polar)

Let's check an example to clarify the above mentioned points



A pendulum is attached with an linearly oscillating particle

❑ (x, θ) are the independent generalized coordinates. (Check the independence)

❑ Generalized coordinates

x →distance

θ →Angle

Not belong to any specific coordinates system (mixed up)

Generalized coordinates properties

- **q_j →To be generalized coordinates**

They must be

- Must be independent
- Must be complete
- System must be holonomic

- **Meaning of Complete**: Capable to describe the system configuration at times. In other word, capable of locating all parts at all times.

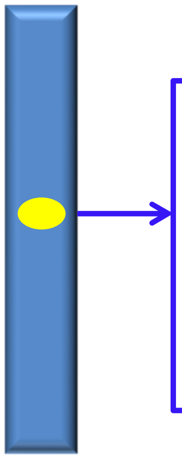
- Generalized coordinates
  - Not necessarily Cartesian
  - Not necessarily any specific coordinate system

Generalized coordinates of rigid body

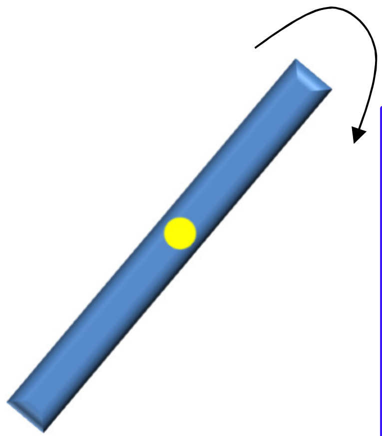
❑**Rigid body has six degrees of freedom**

Thus **six generalized coordinates** are necessary to specify the dynamics of rigid body

3 translational DOF for the center of Mass + 3 rotational degree of freedom about the center of mass = 6 generalized coordinates

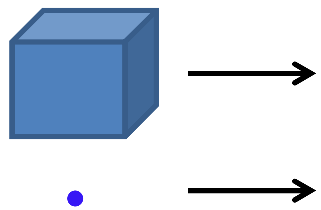


Translational degree of freedom of CM : (x, y, z)



Three rotational degree of freedom about CM: (φ, θ, ψ)

In case of only translation (motion of CM), a rigid body can be accounted as point particle during estimating the number degree of freedom



Summery

Constrain

- **Holonomic**
  $f(q_i, t) = 0$
- **Non-Holonomic**
  $f(q_i, \dot{q}_i, t) = 0$
  Or, as inequalities
  $f(q_i) \geq 0$

- Degree's of freedom =No. of **independent** coordinates required to completely specify particles configuration at all times (generalized coordinates) $=3N - k$

Where N→ no. of particles
k → no. of holonomic, constrains

- Choice of generalized coordinates is not unique but no. must be equal to degree's of freedom.

Question please