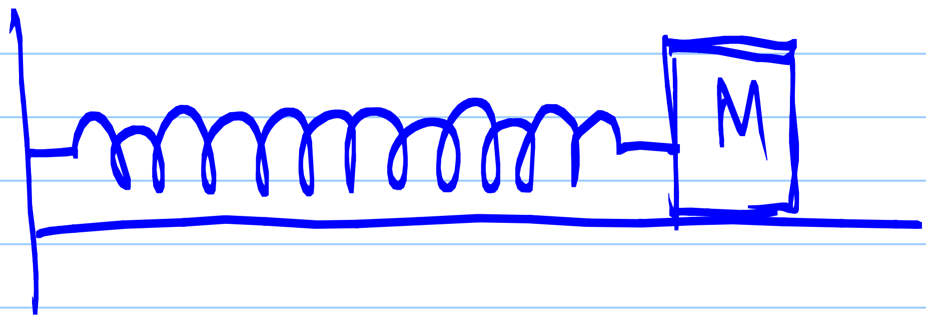
PH 101 Lecture - 05 Aug 09, 2012 DKG

Note Title 8/7/2012

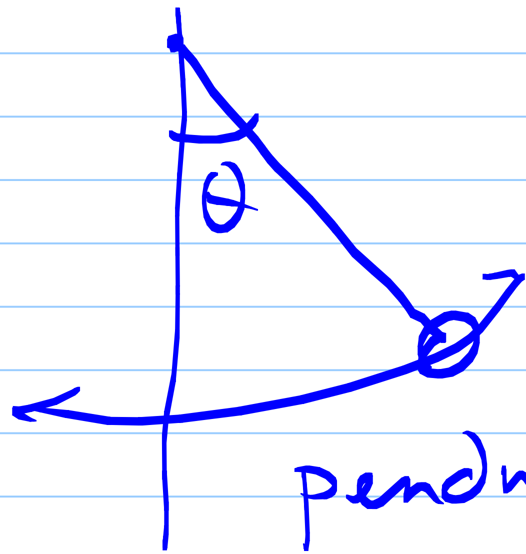
Two examples (Simple harmonic motion)

1. Spring & ball

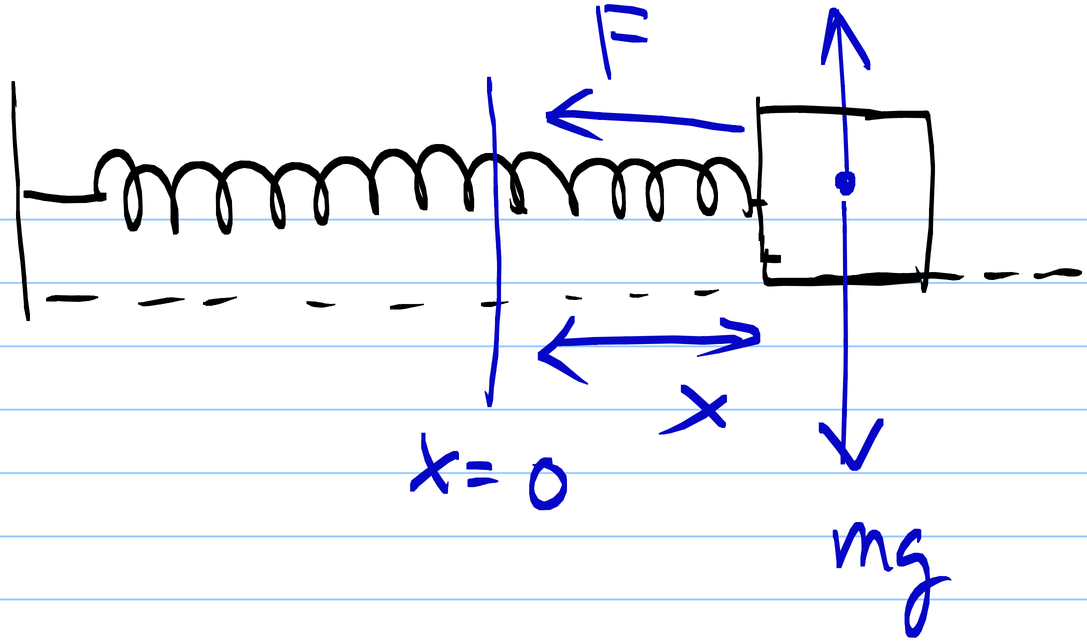
2. Pendulum.



Spring & ball



pendulum



$|F| \propto |x|$

$F = -kx$

Restoring force

$k \rightarrow$ Spring constant

Allow oscillation:

$m\ddot{x} = -kx$

$m\ddot{x} + kx = 0$

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

Period $T = \dfrac{2\pi}{\omega}$ (sec)

$f = \dfrac{1}{T}$ Hz.

$$x(t) = A\cos(\omega t + \phi)$$

Amplitude

Angular frequency

phase

Calculate $\dot{x}(t) = -A\omega \sin(\omega t + \phi)$

$$\ddot{x}(t) = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x(t)$$

$$-\omega^2 x + \frac{k}{m} x = 0$$

$$\omega^2 = \frac{k}{m}, \quad \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

! Independent of amplitude, phase

Phase $\rightarrow$ Initial conditions !

* fixed position
* initial velocity

Example

$x = 0, \quad t = 0 \leftarrow$ equilibrium!

$V = -3\ m/s$

$k = 10\ N/m, \quad m = 0.1\ kg$

$\omega = \sqrt{\frac{k}{m}} = 10\ rad/s, \quad T = 0.628\ Sec$

$f = 1.6\ Hz.$

Initial Condition!

$x(t) = 0 = A\cos(\phi)$

$\cos\phi = 0$

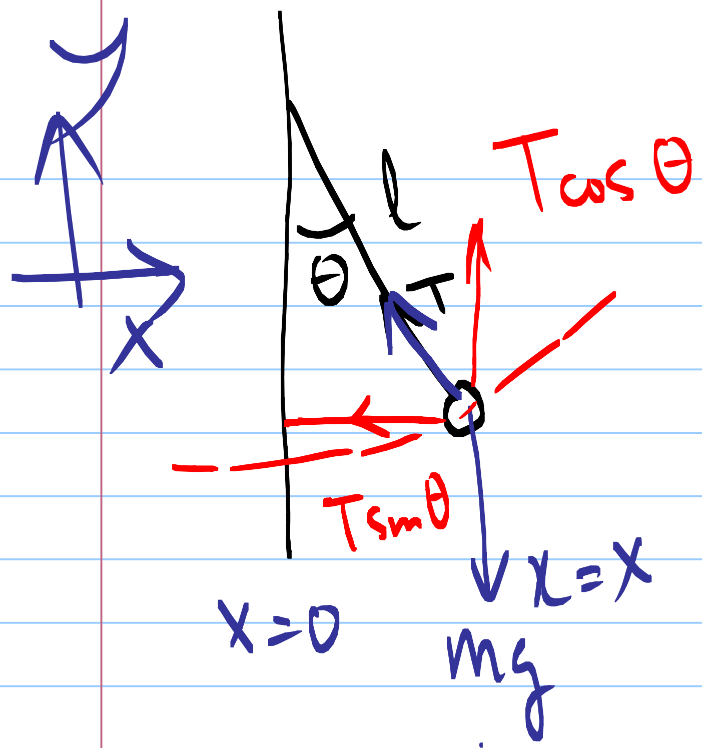
$\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

$\dot{x} = -3 = -A \cdot 10 \cdot \sin\phi$

$\phi = \pi/2 \rightarrow A = 0.3$

$\phi = \frac{3\pi}{2} \rightarrow A = -0.3$

$x(t) = \pm 0.3\cos(10t + \frac{\pi}{2})$



Eq^n of motion!

x: $\quad m\ddot{x} = -T(\theta)\sin\theta = -T(\theta)\frac{x}{l}$

restoring force!

y: $\quad m\ddot{y} = T(\theta)\cos\theta - mg$

Take small angle approximation:

① $\theta \ll 1$ rad — small

$\cos\theta \sim 1$

$$\left.\begin{aligned}\cos 5^\circ &= 0.996\\ \cos 10^\circ &= 0.985\end{aligned}\right\}$$

② Displacement along y — small!

$\ddot{y} = 0$

$$0 = T - mg \Rightarrow T = mg$$

$$m\ddot{x} + \frac{mg}{l}x = 0$$

$$\boxed{\ddot{x} + \frac{g}{l}x = 0}$$

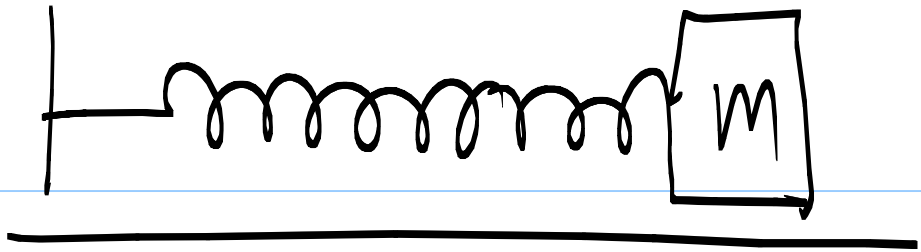
<u>SHO</u>

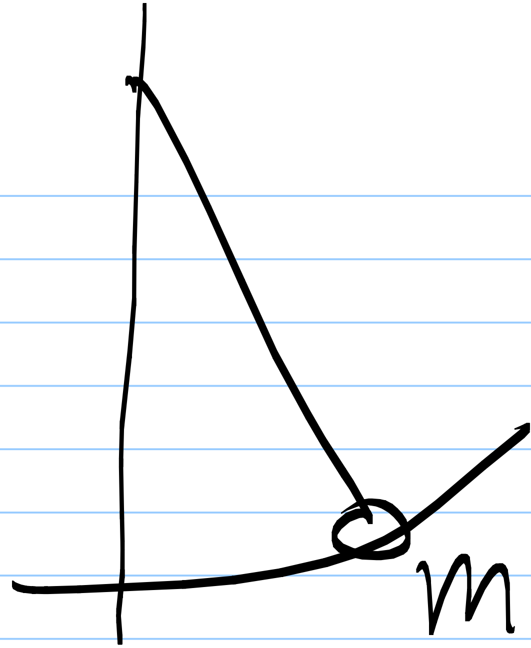
sol^n $x(t) = A\cos(\omega t + \phi)$

$$\omega = \sqrt{\frac{g}{l}}, \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Depends on l & g

does not depend on mass





$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

"Momentum"

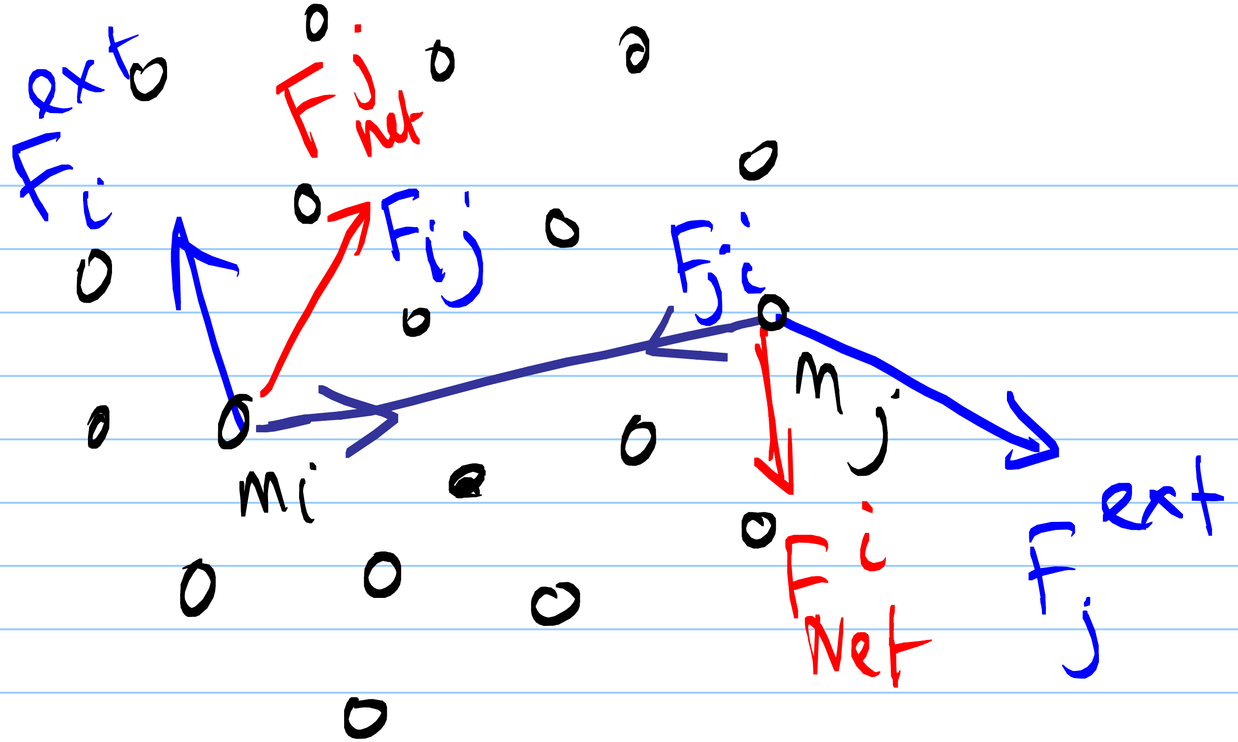
Momentum $\vec{p} = m\vec{v}$ $\quad kg\ m/s$

$$\vec{F} = m\vec{a} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

mass constant

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$ — Newton's 2nd law!

momentum $\vec{p}$ is more fundamental than velocity!



$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_i + \cdots$$

$$\frac{d\vec{p}_{tot}}{dt} = \vec{F}_{Net}^{1} + \vec{F}_{Net}^{2} + \vec{F}_{Net}^{3} + \cdots + \vec{F}_{Net}^{i} + \cdots$$

$$= \vec{F}_{tot} = \vec{F}_{int} + \vec{F}_{ext}$$

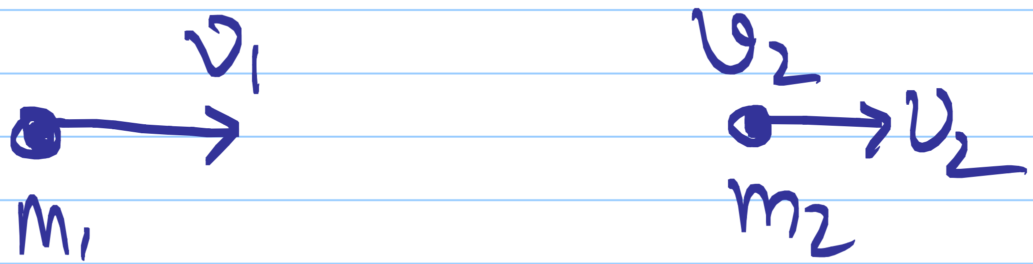
$\vec{F}_{int} = 0$ [by 3rd law]

$$\frac{d\vec{p}_{tot}}{dt} = \vec{F}_{ext}$$

if $\vec{F}_{ext} = 0$ Momentum of the System is CONSERVED

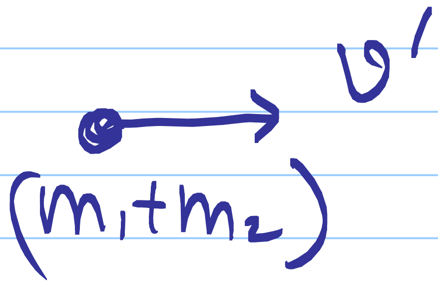
Example:

1D



Before Collision

$$\vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2$$



After Collision

$$\vec{p} = (m_1+m_2)v'$$

$m_1 = 1\ \text{kg}$ | $v_1 = 5\ m/s$

$m_2 = 2\ \text{kg}$ | $v_2 = 3\ m/s$

Momentum Conservation $\Rightarrow$

$$5 \times 1 + 2 \times 3 = 11 = (1+2)\, v'$$

$$\Rightarrow \quad v' = \frac{11}{3} \quad m/s$$

What happens to K.E. ??

K.E. Before collision $= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = 21.5\ J$

K.E. After " $= \frac{1}{2}(m_1 + m_2) v'^2 = 20.2\ J$

$m_1 = 1\,kg$ $\qquad$ $m_2 = 2\,kg$ $\qquad$ Befor Collision

$v_1 = +5\ m/s$ $\qquad$ $v_2 = -3\ m/s$

$$\vec{p}_{net} = 5\times1 - 3\times2 = -1\ kg\,m/s$$

v'

$$(m_1+m_2)\,v_1' = -1$$

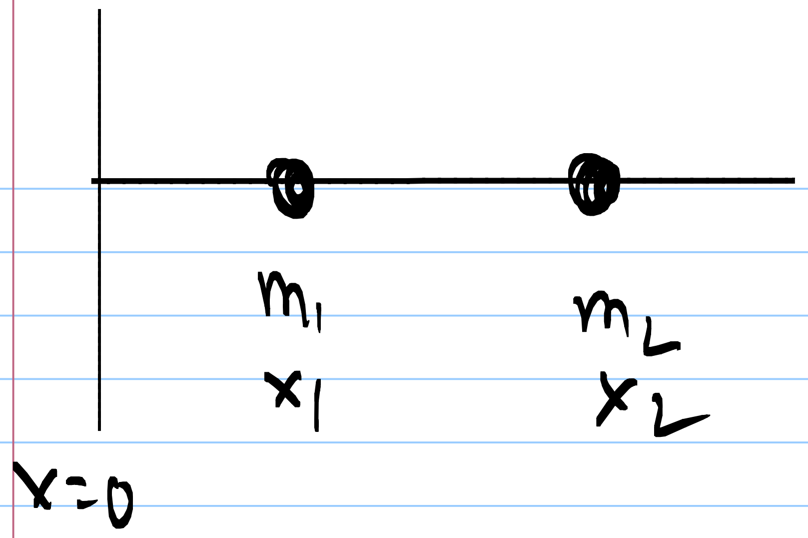
$$v_1' = -\frac{1}{3}\ m/s$$

K.E After Collision $= \frac{1}{2}\,3\times\left(\frac{1}{3}\right)^2 = \frac{1}{6} = 0.17\ J$

K.E Destroyed !

5kg 1m/s $\qquad$ 5kg 1m/s $\quad \Longrightarrow \quad$ $V = 0$

$\boxed{K.E = 0}$



$$m_1\ddot{\vec{x}}_1 = \vec{F}_{12} + \vec{F}^1_{ext}$$

$$m_2\ddot{\vec{x}}_2 = \vec{F}_{21} + \vec{F}^2_{ext}$$

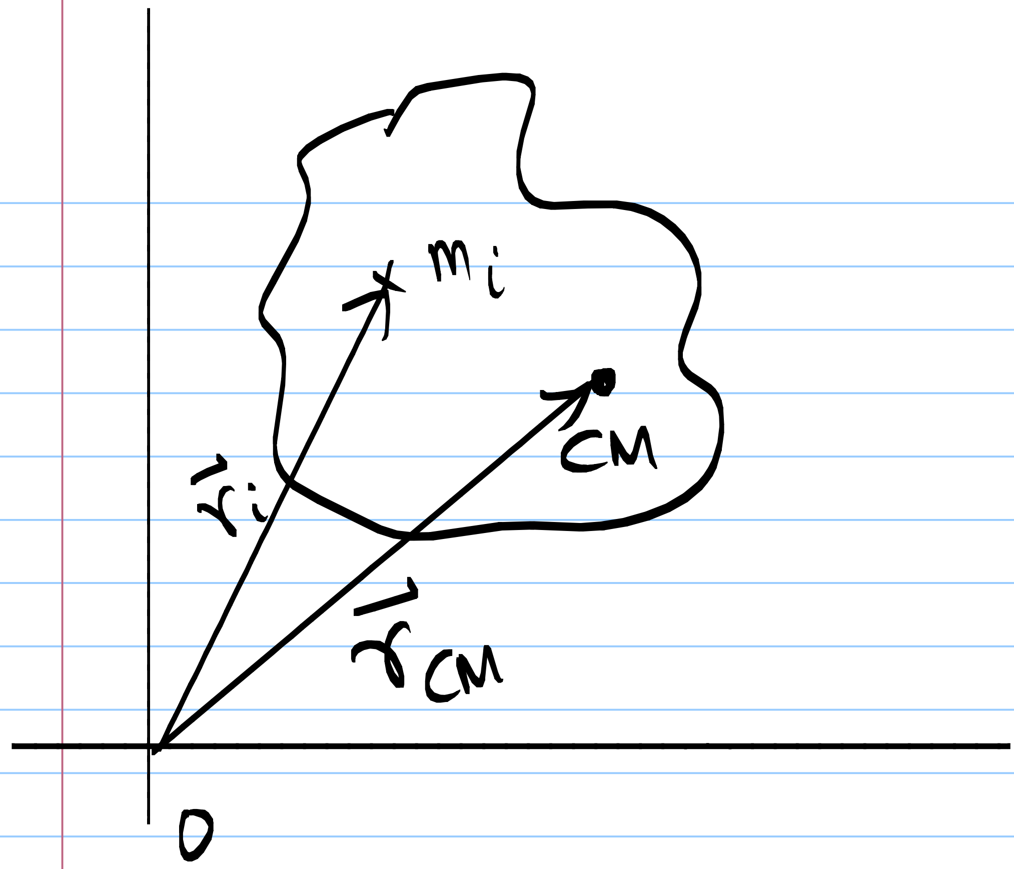
$$m_1\ddot{\vec{x}}_1 + m_2\ddot{\vec{x}}_2 = \vec{F}_{ext}$$

$$(m_1+m_2)\frac{m_1\ddot{\vec{x}}_1 + m_2\ddot{\vec{x}}_2}{m_1+m_2} = \vec{F}_{ext}$$

$$M\ddot{\vec{X}}_{CM} = \vec{F}_{ext}$$

$$X_{CM} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

Center of mass!



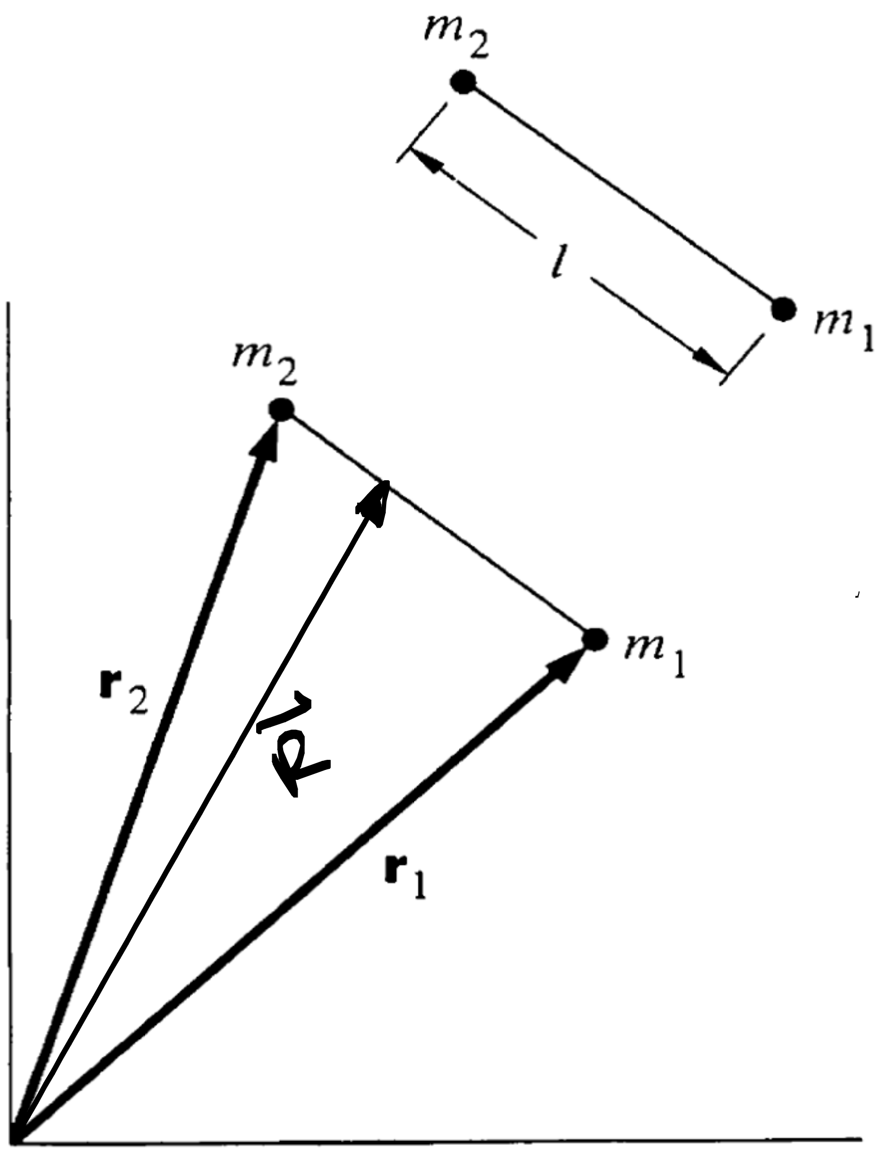
$$M_{tot}\,\vec{r}_{CM} = \sum_i m_i \vec{r}_i$$

$$\vec{v}_{CM} = \frac{1}{M_{tot}} \sum_i m_i \vec{v}_i$$

$$= \frac{1}{M_{tot}} \vec{p}_{tot}$$

$$\vec{p}_{tot} = M_{tot}\,\vec{v}_{CM}$$

$$\frac{d\vec{p}_{tot}}{dt} = M_{tot} \frac{d\vec{v}_{CM}}{dt} = M_{tot}\,\vec{a}_{CM}$$



Center of mass

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Neglect the mass of thin rod!

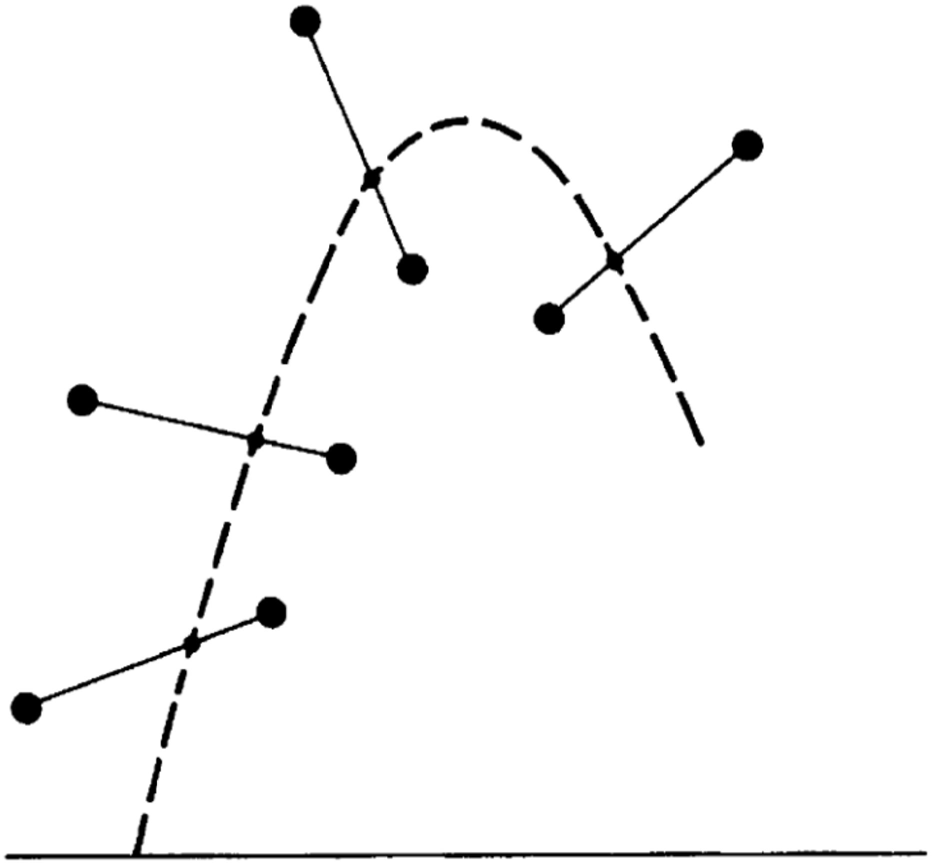
Baton is thrown into air!

What is eqn of motion?

External force on the baton is

$$\vec{F} = m_1\vec{g} + m_2\vec{g}$$

$$(m_1 + m_2)\ddot{\vec{R}} = (m_1 + m_2)\, g$$



$$\ddot{\vec{R}} = \vec{g}$$

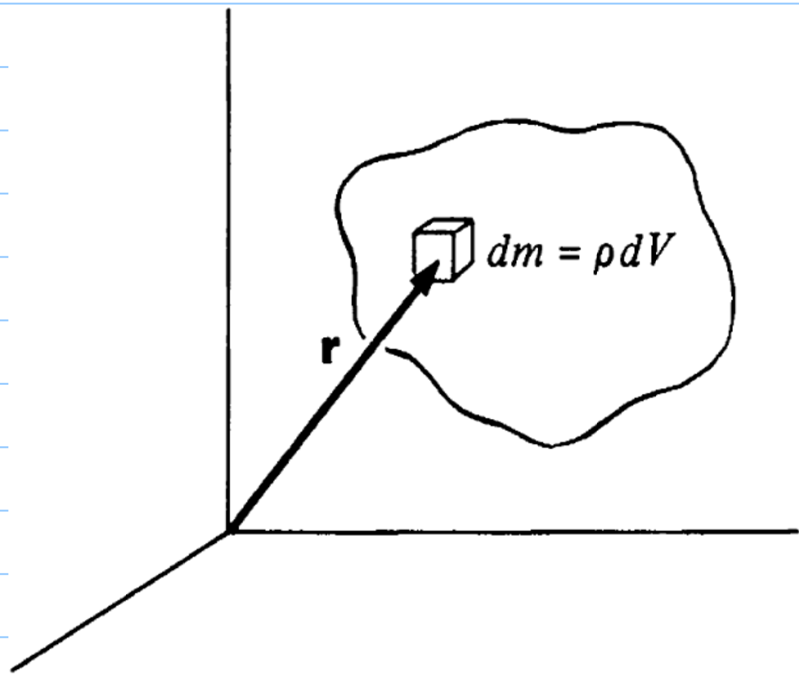
$\rightarrow$ motion under gravitation force of a single object.

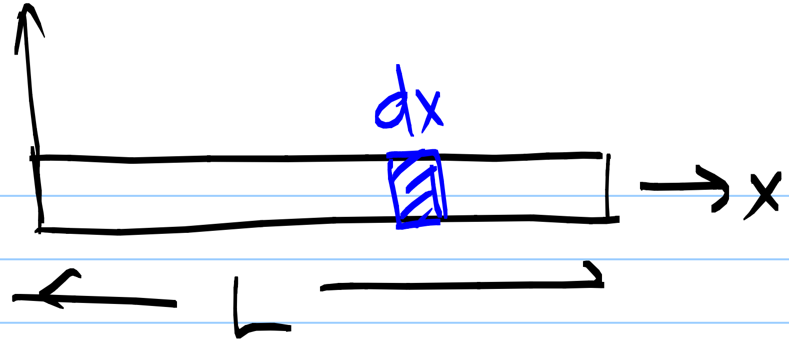
Center of mass of an extended object

$$\vec{R} = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j$$

$N \to \infty$ for extended object

$$\vec{R} = \frac{1}{M} \int \vec{r} \, dm$$



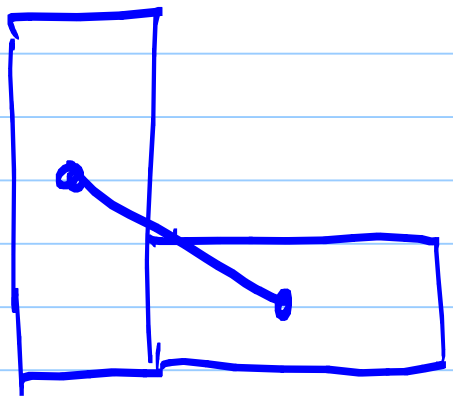


mass M

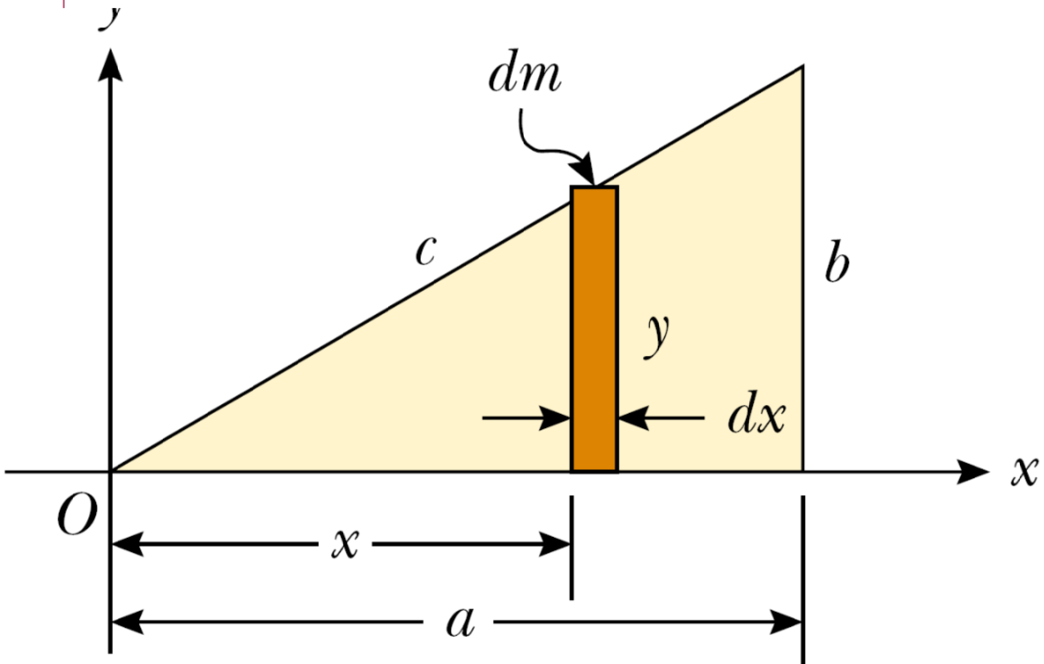
Length L

$$R = \frac{\int x\,dm}{M} = \frac{\int x \frac{M}{L}\,dx}{M}$$

$$= \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\cdot\frac{L^2}{2} = \frac{L}{2}$$



Center of mass of a triangular Sheet



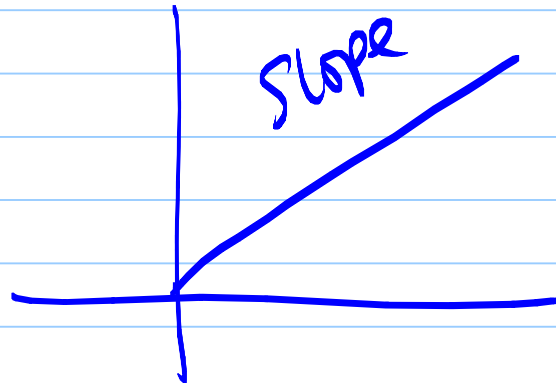
$$dm = \frac{\text{total mass}}{\text{total area}} \times \text{strip area}$$

$$= \frac{M}{\frac{1}{2}ab}(y\,dx)$$

$$= \frac{2M}{ab}(y\,dx)$$

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\cdot\int_0^a \frac{2M}{ab} x y\,dx = \frac{2}{ab}\int_0^a xy\,dx$$

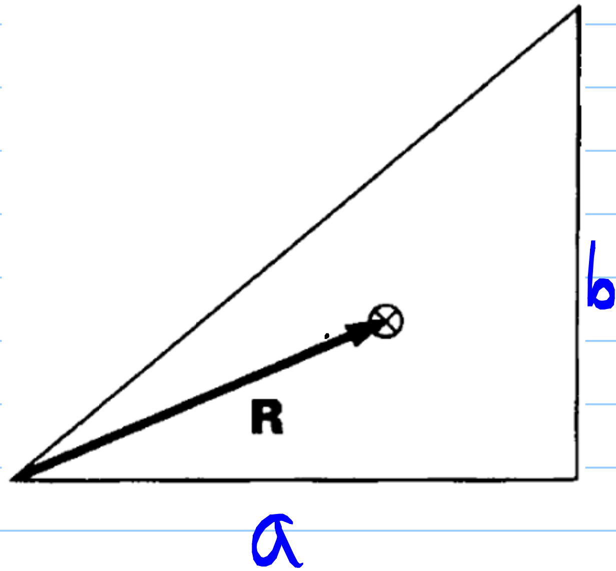
$$\frac{y}{x} = \frac{b}{a} \Rightarrow y = \left(\frac{b}{a}\right)x$$ — eq^n of straight line



$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right)dx$$

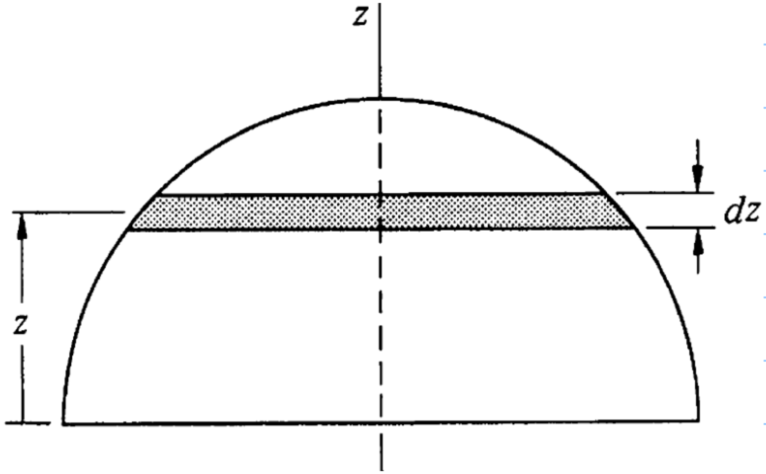
$$= \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{3}a$$

$$Y_{CM} = \frac{1}{3}b$$



$$\boxed{\vec{R} = \frac{2}{3}a\,\hat{i} + \frac{1}{3}b\,\hat{j}}$$

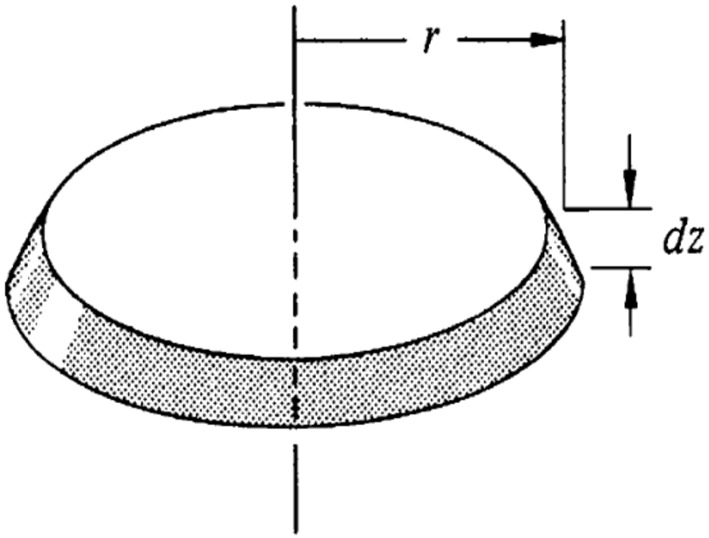
Center of mass of a hemisphere of radius R and mass M



$$Z = \frac{1}{M}\int z\,dM$$

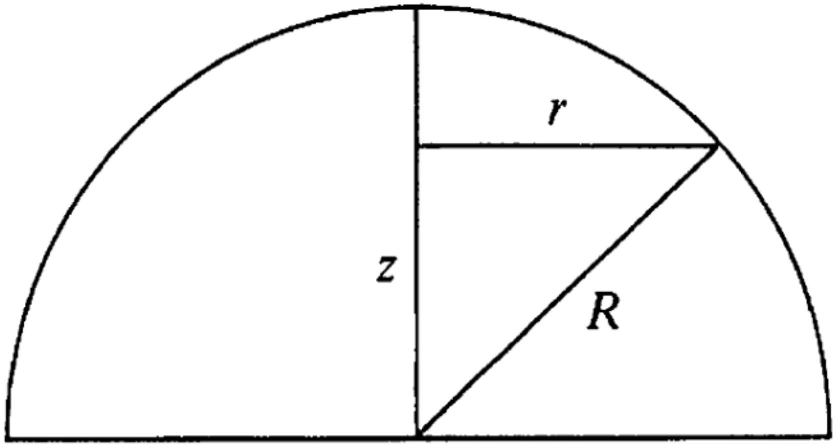
$$dM = \rho\,dV = \frac{M}{V}\,dV$$

$$V = \frac{2}{3}\pi R^3$$



$$Z = \frac{1}{M}\int \frac{M}{V} z\,dV$$

$$= \frac{1}{V}\int_{z=0}^{R} \pi r^2 z\,dz$$



$$r^2 = R^2 - z^2$$

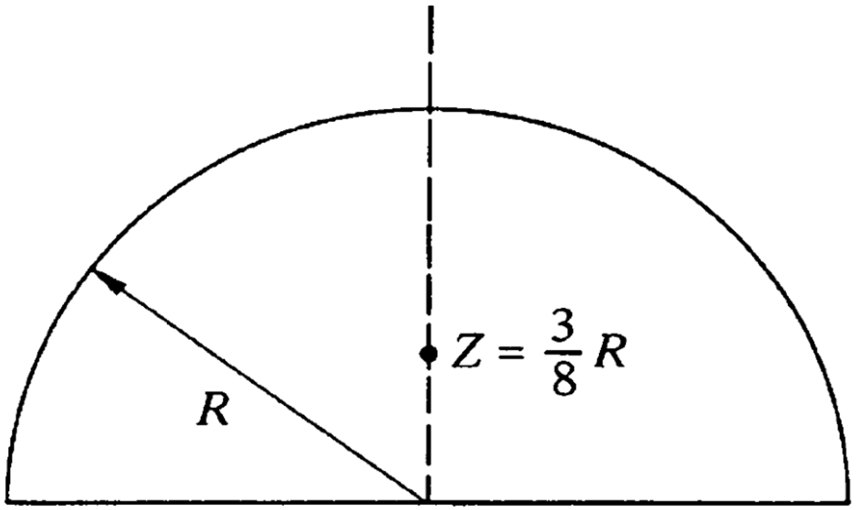
$$Z = \frac{\pi}{V}\int_0^R z(R^2 - z^2)\,dz$$

$$= \frac{\pi}{V}\left(\frac{1}{2}z^2R^2 - \frac{1}{4}z^4\right)\Big|_0^R$$

$$= \frac{\pi}{V}\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right)$$

$$= \frac{\frac{1}{4}\pi R^4}{\frac{2}{3}\pi R^3}$$

$$= \frac{3}{8}R.$$



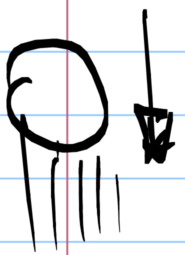
Impulse !

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Differential form !

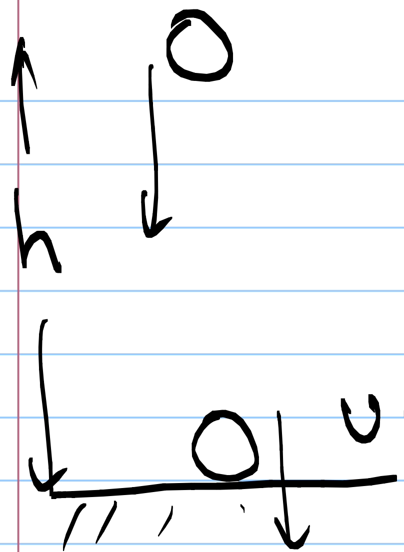
$$\underbrace{\int_0^t \vec{F}\,dt}_{\text{Impulse}} = \underbrace{\vec{p}(t) - \vec{p}(0)}_{\text{Change in momentum}}$$

Integral form !



Small force — large time interval

large force — short time interval



Impact time $\simeq 2\,\text{ms}$

elastic Collision

$v=\sqrt{2gh}$ $\quad mv\downarrow \quad mv\uparrow \equiv 2mv = I$

weight

1 N

$m = 0.1\ \text{kg}$

$h = 1.5\ \text{m}$

$v = 5.5\ \text{m/s}$

$I = 2mv = 1.1\ \text{kg m/s}$

$$\langle F\rangle = \frac{I}{\Delta t} = \frac{1.1\ \text{kg m/s}}{2\,\text{ms}} = 550\ \text{N}$$

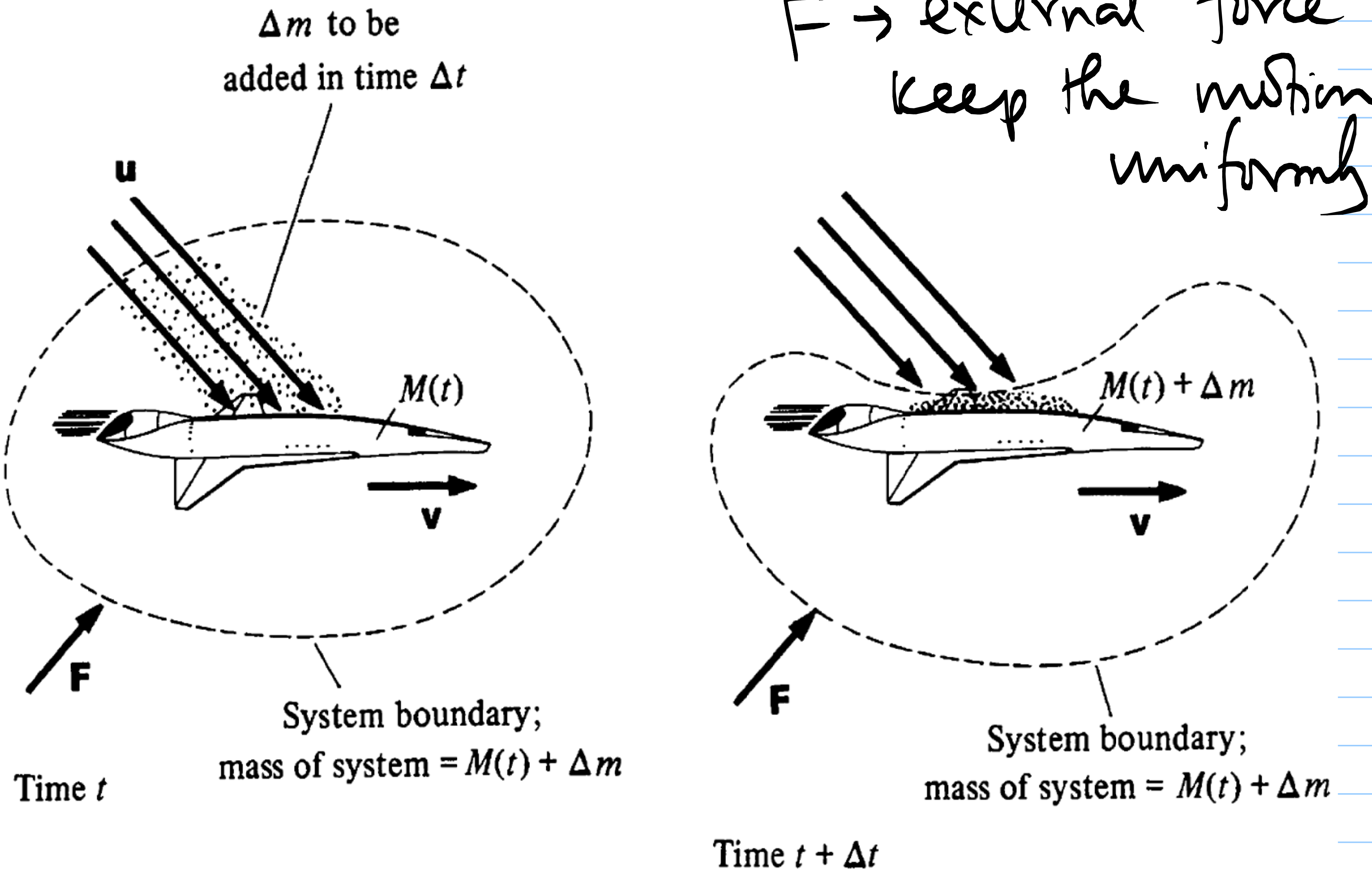
Momentum and flow of mass:

$$\vec{F} = \frac{d\vec{P}}{dt} \longrightarrow$$ Considered a set of particles

$$I = \int_{t_a}^{t_b} \vec{F}\, dt = P(t_b) - P(t_a)$$

keep track of all the particles within time interval t_a & t_b

F → external force to keep the motion uniformly



System consists of $M(t)$ and Δm

$$\vec{P}(t) = M(t)\vec{v} + (\Delta m)\vec{u}$$ — initial momentum

$$P(t+\Delta t) = M(t)\,\vec{v} + (\Delta m)'\,\vec{v}$$ — final momentum

Change in momentum

$$\Delta\vec{p} = P(t+\Delta t) - P(t)$$

$$= (\vec{v}-\vec{u})\,\Delta m$$

$$\frac{\Delta\vec{p}}{\Delta t} = (\vec{v}-\vec{u})\frac{\Delta m}{\Delta t} \quad \text{at} \quad \Delta t \to 0$$

$$\frac{d\vec{p}}{dt} = (\vec{v}-\vec{u})\frac{dm}{dt} = \vec{F}$$

F can be either +ve or −ve. → depends on u & v

if $u = v \Rightarrow F = 0 \rightarrow$ momentum constant.

Look back

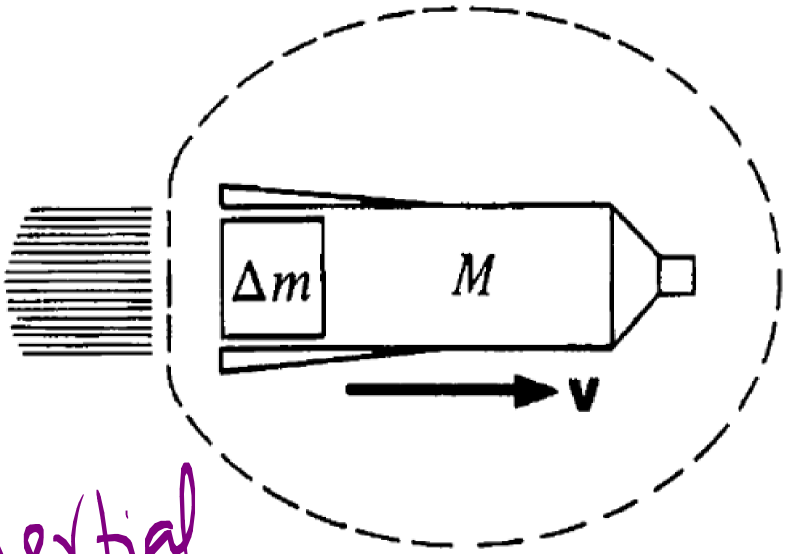
$$F = \frac{d}{dt}(mv)$$

$$= \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

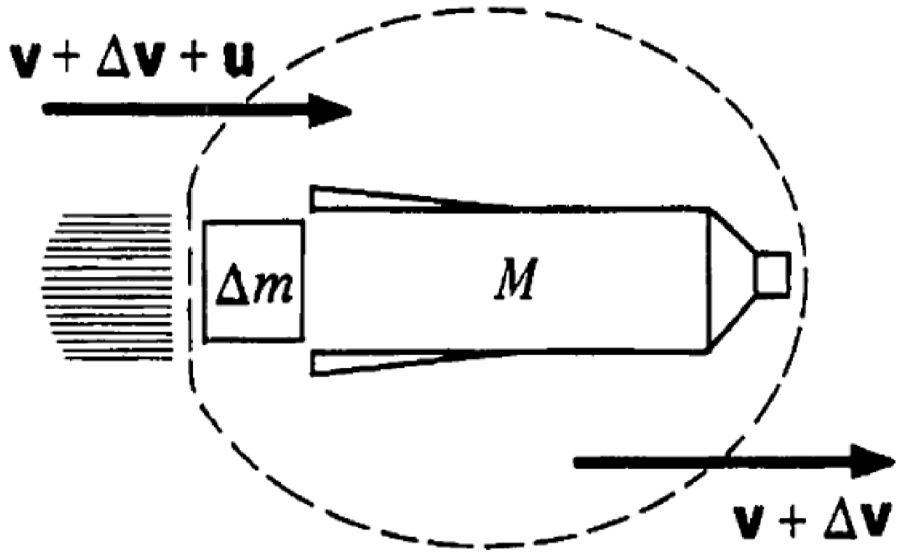
$$v \to \text{constant} \Rightarrow \frac{dv}{dt} = 0$$

$$\vec{F} = \vec{v}\,\frac{dm}{dt}$$

However,
$$\vec{F} = (\vec{v} - \vec{u})\,\frac{dm}{dt}$$



Time t



Time $t + \Delta t$

Inertial frame

$$P(t) = (M + \Delta m)\vec{v}$$

$$P(t+\Delta t) = M(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \Delta\vec{v} + \vec{u})$$

Change in momentum is

$$\Delta\vec{p} = \vec{P}(t+\Delta t) - \vec{P}(t)$$

$$= M\,\Delta\vec{v} + (\Delta m)\,\vec{u}$$

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt}$$

Now $$\frac{dm}{dt} = -\frac{dM}{dt}$$

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}}$$

<u>HW</u>

- No external force $\rightarrow$ $\vec{F}=0$ Rocket in free space!
- Under Gravity $\vec{F} = Mg$

When $\vec{F}=0$, No external force!

$$M\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dM}{dt}$$

or

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt}.$$

$$\int_{t_0}^{t_f}\frac{d\mathbf{v}}{dt}\,dt = \mathbf{u}\int_{t_0}^{t_f}\frac{1}{M}\frac{dM}{dt}\,dt$$

$$= \mathbf{u}\int_{M_0}^{M_f}\frac{dM}{M}$$

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\frac{M_f}{M_0}$$

$$= -\mathbf{u}\ln\frac{M_0}{M_f}.$$

Rocket equation!

Under gravity: $\vec{F} = m\vec{g}$

$$M\vec{g} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}$$

$\vec{u}$ and $\vec{g}$ are in same direction:

$$\frac{d\vec{v}}{dt} = \frac{u}{M}\frac{dM}{dt} + g$$

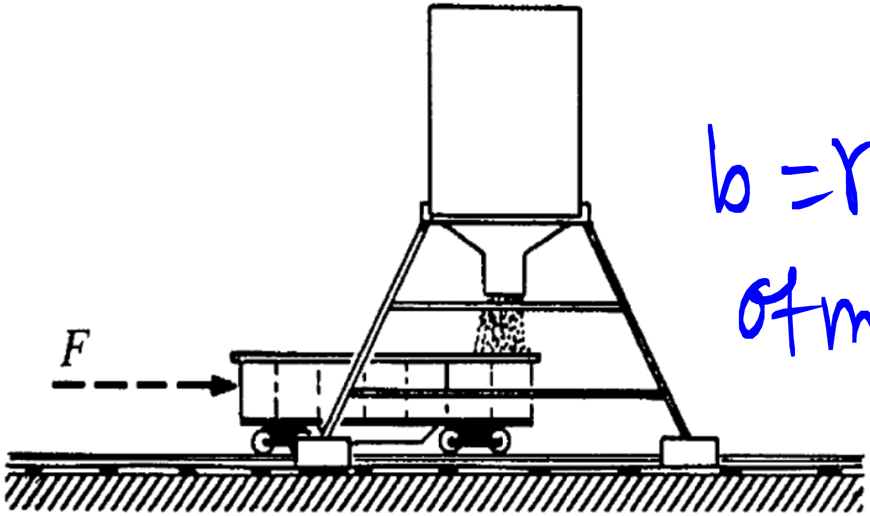
Integrating:

$t_f \rightarrow$ Burn time

$$v_f - v_0 = u \ln\frac{M_f}{M_0} + g(t_f - t_0)$$

at $V_0 = 0$, $t_0 = 0$ | $v_t = u \ln\left(\frac{M_0}{M_f}\right) - g t_f$

Initially car is at rest!



b = rate of mass flow

$$\vec{F}t = \vec{P}_f - \vec{P}_i$$

$$\vec{P}_f = (M + bt)v$$

$$\vec{P}_i = 0$$

$m = bt$

$t = \frac{m}{b}$

$$(M + bt)v = Ft$$

$$v = \frac{Ft}{(M+bt)}$$