Complex Analysis

PH 503 Course™

Charudatt Kadolkar
Indian Institute of Technology, Guwahati

Preface

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Acknowledgments

As of now none but myself

IIT Guwahati *Charudatt Kadolkar*.

Contents

1 Complex Numbers

De• nitions

De• nition 1.1 *Complex numbers are de• ned as ordered pairs* (x, y)

Points on a complex plane. Real axis, imaginary axis, purely imaginary numbers. Real and imaginary parts of complex number. Equality of two complex numbers.

De• nition 1.2 *The sum and product of two complex numbers are de• ned as follows:*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

In the rest of the chapter use $z, z_1, z_2, \ldots$ for complex numbers and x, y for real numbers. introduce i and $z = x + iy$ notation.

Algebraic Properties

1. Commutativity
$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1$$
2. Associativity
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3)$$
3. Distributive Law
$$z(z_1 + z_2) = zz_1 + zz_2$$
4. Additive and Multiplicative Indentity
$$z + 0 = z, \qquad z \cdot 1 = z$$
5. Additive and Multiplicative Inverse
$$\begin{aligned}-z &= (-x, -y) \\ z^{-1} &= \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right), \qquad z \neq 0\end{aligned}$$

6. Subtraction and Division
$$z_1 - z_2 = z_1 + (-z_2)\,, \qquad \frac{z_1}{z_2} = z_1 z_2^{-1}$$

7. Modulus or Absolute Value
$$|z| = \sqrt{x^2 + y^2}$$

8. Conjugates and properties
$$\begin{aligned} \overline{z} &= x - iy = (x, -y) \\ \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\overline{z_1}}{\overline{z_2}} \end{aligned}$$

9.
$$\begin{aligned} |z|^2 &= z\overline{z} \\ \operatorname{Re} z &= \frac{z + \overline{z}}{2}, \operatorname{Im} z = \frac{z - \overline{z}}{2i} \end{aligned}$$

10. Triangle Inequality
$$|z_1 + z_2| \le |z_1| + |z_2|$$

Polar Coordinates and Euler Formula

1. Polar Form: for $z \neq 0$,
$$z = r\,(\cos\theta + i\sin\theta)$$
where $r = |z|$ and $\tan\theta = y/x$. θ is called the argument of z. Since $\theta + 2n\pi$ is also an argument of z,the principle value of argument of z is take such that $-\pi < \theta \le \pi$. For $z = 0$ the $\arg z$ is unde• ned.

2. Euler formula: Symbollically,
$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

3.
$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ z^n &= r^n e^{n\theta} \end{aligned}$$

4. de Moivre's Formula
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Roots of Complex Numbers

Let $z = re^{i\theta}$ then

$$z^{1/n} = r^{1/n} \exp\left(i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$

There are only n distinct roots which can be given by $k = 0, 1, \ldots, n-1$. If θ is a principle value of $\arg z$ then θ/n is called the principle root.

Example 1.1 *The three possible roots of* $\left(\frac{1+i}{\sqrt{2}}\right)^{1/3} = \left(e^{i\pi/4}\right)^{1/3}$ *are* $e^{i\pi/12}, e^{i\pi/12+i2\pi/3}, e^{i\pi/12+i4\pi/3}$.

Regions in Complex Plane

1. $\epsilon - nbd$ of z_0 is de• ned as a set of all points z which satisfy
$$|z - z_0| < \epsilon$$
2. $Deleted\ nbd$ of z_0 is a nbd of z_0 excluding point z_0.
3. Interior Point, Exterior Point, Boundary Point, Open set and closed set.
4. Domain, Region, Bounded sets, Limit Points.

2 Functions of Complex Variables

Functions of a Complex Variable

A *function* f de• ned on a set S is a rule that uniquely associates to each point z of S a a complex number w. Set S is called the *domain* of f and w is called the value of f at z and is denoted by $f(z) = w$.

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) = F(r,\theta)e^{i\Theta(r,\theta)}$

Example 2.1 *Write $f(z) = 1/z^2$ in $u+iv$ form.*

$u(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ *and* $v(x,y) = \frac{2xy}{(x^2+y^2)^2}$

$u(r,\theta) = r^{-2}\cos 2\theta$ *and* $v(r,\theta) = -r^{-2}\sin 2\theta$

Domain of f is $C - \{0\}$.

A *multiple-valued function* is a rule that assigns more than one value to each point of domain.

Example 2.2 $f(z) = \sqrt{z}$*. This function assigns two distinct values to each* $z\ (\neq 0)$*. One can choose the function to be single-valued by specifying*

$$\sqrt{z} = +\sqrt{r}e^{i\theta/2}$$

where θ is the principal value.

Elementary Functions

1. **Polynomials**
$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$
whrere the coef• cients are real. Rational Functions.

2. **Exponential Function**
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$
Converges for all z. For real z the de• nition coincides with usual exponential function. Easy to see that $e^{i\theta} = \cos\theta + i\sin\theta$. Then
$$e^z = e^x(\cos y + i \sin y)$$
   a. $e^{z_1} e^{z_2} = e^{z_1 + z_2}$.
   b. $e^{z+2\pi i} = e^z$.
   c. A line segment from $(x, 0)$ to $(x, 2\pi)$ maps to a circle of radius e^x centered at origin.
   d. No Zeros.

3. **Trigonometric Functions**
De• ne
$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2} \\ \tan z &= \frac{\sin z}{\cos z} \end{aligned}$$
   a. $\sin^2 z + \cos^2 z = 1$
   b. $2 \sin z_1 \cos z_2 = \sin(z_1 + z_2) + \sin(z_1 - z_2)$
   c. $2 \cos z_1 \cos z_2 = \cos(z_1 + z_2) + \cos(z_1 - z_2)$
   d. $2 \sin z_1 \sin z_2 = -\cos(z_1 + z_2) + \cos(z_1 - z_2)$
   e. $\sin(z + 2\pi) = \sin z$ and $\cos(z + 2\pi) = \cos z$.
   f. $\sin z = 0$ iff $z = n\pi \qquad (n = 0, \pm 1, \ldots)$
   g. $\cos z = 0$ iff $z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \ldots)$
   h. These functions are not bounded.
   i. A line segment from $(0, y)$ to $(2\pi, y)$ maps to an ellipse with semimajor axis equal to $\cosh y$ under $\sin$ function.

4. **Hyperbolic Functions**
De• ne
$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} \\ \sinh z &= \frac{e^z - e^{-z}}{2} \end{aligned}$$

a. $\sinh(iz) = i \sin z$; $\cosh(iz) = \cos z$
b. $\cosh^2 z - \sinh^2 z = 1$
c. $\sinh(z + 2\pi i) = \sinh(z)$; $\cosh(z + 2\pi i) = \cosh(z)$
d. $\sinh z = 0$ iff $z = n\pi i \quad (n = 0, \pm 1, \ldots)$
e. $\cosh z = 0$ iff $z = \left(\frac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \ldots)$

5. **Logarithmic Function**

   De• ne
   $$\log z = \log r + i(\theta + 2n\pi)$$
   then $e^{\log z} = z$.

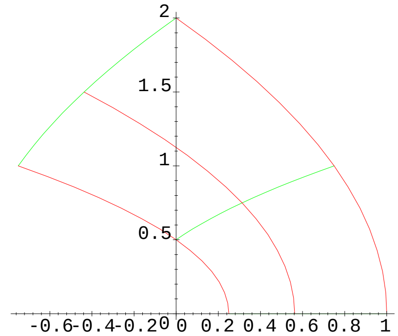
   a. Is multiple-valued. Hence cannot be considered as inverse of exponetial function.
   b. Priniciple value of log function is given by
   $$\log z = \log r + i\Theta$$
   where Θ is the principal value of argument of z.
   c. $\log(z_1 z_2) = \log z_1 + \log z_2$

Mappings

$w = f(z)$. Graphical representation of images of sets under f is called *mapping*. Typically shown in following manner:

1. Draw regular sets (lines, circles, geometric regions etc) in a complex plane, which we call z plane. Use $z = x + iy = re^{i\theta}$.
2. Show its images on another complex plane, which we call w plane. Use $w = f(z) = u + iv = \rho e^{i\phi}$.

Example 2.3 $w = z^2$.



Mapping of z^2 *Mapping* z^2

1. A straight line $x = t$ maps to a parabola $v^2 = -4t^2 \left(u - t^2\right)$
2. A straight line $y = t$ maps to a parabola $v^2 = 4t^2 \left(u - t^2\right)$
3. A half circle given by $z = r_0 e^{i\theta}$ where $0 \leq \theta \leq \pi$ maps to a full circle given by $w = r_0^2 e^{i2\theta}$. This also means that the upper half plane maps on to the entire complex plane.
4. A hyperbola $x^2 - y^2 = c$ maps to a straight line $u = c$.

Mappings by Elementary Functions.

1. **Translation** by z_0 is given by $w = z + z_0$.
2. **Rotation** through an angle θ_0 is given by $w = e^{i\theta} z$.
3. **Rel• ection** through x axis is given by $w = \bar{z}$.
4. **Exponential Function**

Exponential Function

A vertical line maps to a circle.

A horizontal line maps to a radial line.

A horizontal strip enclosed between $y = 0$ and $y = 2\pi$ maps to the entire complex plane.

5. **Sine Function** $\sin z = \sin x \cosh y + i \cos x \sinh y$

A vertical line maps to a branch of a hyperbola.

A horizontal line maps to an ellipse and has a period of 2π.

3 Analytic Functions

Limits

A function f is de• ned in a deleted nbd of z_0.

De• nition 3.1 *The limit of the function $f(z)$ as $z \to z_0$ is a number w_0 if, for any given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Example 3.1 $f(z) = 5z$. *Show that* $\lim_{z \to z_0} f(z) = 5z_0$.

Example 3.2 $f(z) = z^2$. *Show that* $\lim_{z \to z_0} f(z) = z_0^2$.

Example 3.3 $f(z) = z/\bar{z}$. *Show that the limit of f does not exist as $z \to 0$.*

Theorem 3.1 *Let $f(z) = u(x, y) + iv(x, y)$ and $w_0 = u_0 + iv_0$. $\lim\limits_{z \to z_0} f(z) = w_0$ if and only if $\lim\limits_{(x,y) \to (x_0,y_0)} u = u_0$ and $\lim\limits_{(x,y) \to (x_0,y_0)} v = v_0$.*

Example 3.4 $f(z) = \sin z$. *Show that the* $\lim_{z \to z_0} f(z) = \sin z_0$

Example 3.5 $f(z) = 2x + iy^2$. *Show that the* $\lim_{z \to 2i} f(z) = 4i$.

Theorem 3.2 *If* $\lim\limits_{z \to z_0} f(z) = w_0$ *and* $\lim\limits_{z \to z_0} F(z) = W_0$,

$$\begin{aligned} \lim_{z \to z_0} [f(z) + F(z)] &= w_0 + W_0; \\ \lim_{z \to z_0} f(z) F(z) &= w_0 W_0; \\ \lim_{z \to z_0} f(z) / F(z) &= \frac{w_0}{W_0} \qquad W_0 \neq 0. \end{aligned}$$

This theorem immediately makes available the entire machinery and tools used for real analysis to be applied to complex analysis. The rules for • nding limits then can be listed as follows:

1. $\lim_{z \to z_0} c = c$.
2. $\lim_{z \to z_0} z^n = z_0^n$.
3. $\lim_{z \to z_0} P(z) = P(z_0)$ if P is a polynomial in z.
4. $\lim_{z \to z_0} \exp(z) = \exp(z_0)$.
5. $\lim_{z \to z_0} \sin(z) = \sin z_0$.

Continuity

De• nition 3.2 *A function f, de• ned in some nbd of z_0 is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This de• nition clearly assumes that the function is de• ned at z_0 and the limit on the LHS exists. The function f is continuous in a region if it is continuous at all points in that region.

If funtions f and g are continuous at z_0 then $f + g$, fg and f/g $(g(z_0) \neq 0)$ are also continuous at z_0.

If a function $f(z) = u(x, y) + iv(x, y)$ is continuous at z_0 then the component functions u and v are also continuous at (x_0, y_0).

Derivative

De• nition 3.3 *A function f, de• ned in some nbd of z_0 is differentiable at z_0 if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is called the derivative of f at z_0 and is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

Example 3.6 $f(z) = z^2$. *Show that $f'(z) = 2z$.*

Example 3.7 $f(z) = |z|^2$. *Show that this function is differentiable only at $z = \dot{0}$. In real analysis $|x|$ is not differentiable but $|x|^2$ is.*

If a function is differentiable at z, then it is continuous at z.

The converse in not true. See Example 3.7.

Even if component functions of a complex function have all the partial derivatives, does not imply that the complex function will be differentiable. See Example 3.7.

Some rules for obtaining the derivatives of functions are listed here. Let f and g be differentiable at z.

1. $\frac{d}{dz}(f \pm g)(z) = f'(z) \pm g(z)$.
2. $\frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z)$.
3. $\frac{d}{dz}(f/g)(z) = (f'(z)g(z) - f(z)g'(z)) / [g(z)]^2$ if $g(z) \neq 0$.
4. $\frac{d}{dz}(f \circ g)(z) = f'(g(z))g'(z)$.
5. $\frac{d}{dz}c = 0$.
6. $\frac{d}{dz}z^n = nz^{n-1}$.

Cauchy-Riemann Equations

Theorem 3.3 *If $f'(z_0)$ exists, then all the • rst order partial derivatives of component function $u(x,y)$ and $v(x,y)$ exist and satisfy Cauchy-Riemann Conditions:*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Example 3.8 $f(z) = z^2 = x^2 - y^2 + i2xy$. *Show that Cauchy-Riemann Condtions are satis• ed.*

Example 3.9 $f(z) = |z|^2 = x^2 + y^2$. *Show that the Cauchy-Riemann Condtions are satis• ed only at $z = 0$.*

Theorem 3.4 *Let $f(z) = u(x,y) + iv(x,y)$ be de• ned in some nbd of the point z_0. If the • rst partial derivatives of u and v exist and are continuous at z_0 and satisfy Cauchy-Riemann equations at z_0, then f is differentiable at z_0 and*

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Example 3.10 $f(z) = \exp(z)$. *Show that $f'(z) = \exp(z)$.*

Example 3.11 $f(z) = \sin(z)$. *Show that $f'(z) = \cos(z)$.*

Example 3.12 $f(z) = \frac{\bar{z}^2}{z}$. *Show that the CR conditions are satis• ed at $z = 0$ but the function f is not differentiable at 0.*

If we write $z = re^{i\theta}$ then we can write Cauchy-Riemann Conditions in polar coordinates:

$$\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r . \end{aligned}$$

Analytic Functions

Definition 3.4 *A function is analytic in an open set if it has a derivative at each point in that set.*

Definition 3.5 *A function is analytic at a point z_0 if it is analytic in some nbd of z_0.*

Definition 3.6 *A function is an entire function if it is analytic at all points of C.*

Example 3.13 $f(z) = 1/z$ *is analytic at all nonzero points.*

Example 3.14 $f(z) = |z|^2$ *is not analytic anywhere.*

A function is not analytic at a point z_0, but is analytic at some point in each nbd of z_0 then z_0 is called the singular point of the function f.

Harmonic Functions

Definition 3.7 *A real valued function $H(x, y)$ is said to be* harmonic *in a domain of xy plane if it has continuous partial derivatives of the first and second order and satisfies* Laplace equation*:*

$$H_{xx}(x, y) + H_y(x, y) = 0.$$

Theorem 3.5 *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions u and v are harmonic in D.*

Definition 3.8 *If two given functions $u(x, y)$ and $v(x, y)$ are harmonic in domain D and their first order partial derivatives satisfy Cauchy-Riemann Conditions*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x . \end{aligned}$$

then v is said to be harmonic conjugate *of u.*

Example 3.15 *Let $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Show that v is hc of u and not vice versa.*

Example 3.16 $u(x, y) = y^3 - 3xy$. *Find harmonic conjugate of u.*

4 Integrals

Contours

Example 4.1 *Represent a line segment joining points* $(0,6)$ *and* $(3,11)$ *by parametric equations.*

Example 4.2 *Show that a half circle in upper half plane with radius* R *and centered at origin can be parametrized in various ways as given below:*

1. $x(t) = R\cos t, \qquad y(t) = R\sin t$, where $t : 0 \to \pi$.
2. $x(t) = t, \qquad y(t) = \sqrt{R^2 - t^2}$, where $t : -R \to R$.
3. $x(t) = R(2t-1), \qquad y(t) = 2R\sqrt{t - t^2}$, where $t : 0 \to 1$.

De• nition 4.1 *A set of points* $z = (x, y)$ *in complex plane is called an* arc *if*

$$x = x(t), \qquad y = y(t), \qquad (a \le t \le b)$$

where $x(t)$ *and* $y(t)$ *are continuous functions of the real parameter* t.

Example 4.3 $x(t) = \cos t, \qquad y(t) = \sin(2t)$*, where* $t : 0 \to 2\dot{\pi}$*. Show that the curve cuts itself and is closed.*

An arc is called *simple* if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$.

An arc is *closed* if $z(a) = z(b)$.

An arc is *differentiable* if $z'(t) = x'(t) + iy'(t)$ exists and $x'(t)$ and $y'(t)$ are continuous. A *smooth* arc is differentiable and $z'(t)$ is nonzero for all t.

De• nition 4.2 *Length of a smooth arc is de• ned as*

$$L(C) = \int_a^b |\acute{z}'(t)|\, dt.$$

The length is invariant under parametrization change.

Definition 4.3 *A* contour *is a constructed by joining finite smooth curves end to end such that* $z\left(t\right)$ *is continuous and* $z'\left(t\right)$ *is piecewise continuous.*

A closed simple contour has only first and last point same and does not cross itself.

Contour Integral

If C is a contour in complex plane defined by $z\left(t\right) = x\left(t\right) + iy\left(t\right)$ and a function $f\left(z\right) = u\left(x, y\right) + iv\left(x, y\right)$ is defined on it. The integral of $f\left(z\right)$ along the contour C is denoted and defined as follows:

$$\begin{aligned}\int_C f\left(z\right) dz &= \int_a^b f\left(z\right) z'\left(t\right) dt \\ &= \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')dt \\ &= \int (udx - vdy) + i\int (udy + vdx)\end{aligned}$$

The component integrals are usual real integrals and are well defined. In the last form appropriate limits must placed in the integrals.

Some very straightforward rules of integration are given below:

1. $\int_C wf\left(z\right) dz = w\int_C f\left(z\right) dz$ where w is a complex constant.
2. $\int_C (f\left(z\right) + g\left(z\right))dz = \int_C f\left(z\right) dz + \int_C g\left(z\right) dz.$
3. $\int_{C_1+C_2} f\left(z\right) dz = \int_{C_1} f\left(z\right) dz + \int_{C_2} f\left(z\right) dz.$
4. $\left|\int_C f\left(z\right) dz\right| \le \int_C \left|f\left(z\left(t\right)\right) z'\left(t\right)\right| dt.$
5. If $\left|f\left(z\right)\right| \le M$ for all $z \in C$ then $\left|\int_C f\left(z\right) dz\right| \le ML$, where L is length of the countour C.

Example 4.4 $f\left(z\right) = z^2$*. Find integral of* f *from* $(0, 0)$ *to* $(2, 1)$ *along a straight line and also along st line path from* $(0, 0)$ *to* $(2, 0)$ *and from* $(2, 0)$ *to* $(2, 1)$.

Example 4.5 $f\left(z\right) = 1/z$*. Find the integral from* $(2, 0)$ *to* $(-2, 0)$ *along a semicircular path in upper plane given by* $|z| = 2$.

Example 4.6 *Show that* $\left|\int_C f\left(z\right) dz\right| \le \frac{\pi}{3}$ *for* $f\left(z\right) = 1/\left(z^2 - 1\right)$ *and* $C : |z| = 2$ *from* 2 *to* $2i$.

Cauchy-Goursat Theorem

Theorem 4.1 (Jordan Curve Theorem) *Every simple and closed contour in complex plane splits the entire plane into two domains one of which is bounded. The bounded domain is called the interior of the countour and the other one is called the exterior of the contour.*

De• ne a sense direction for a contour.

Theorem 4.2 *Let C be a simple closed contour with positive orientation and let D be the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x and Q_y at all points on C and D, then*

$$\int_C (P(x,y)dx + Q(x,y)\,dy) = \iint [Q_x(x,y) - P_y(x,y)]\,dxdy$$

Theorem 4.3 (Cauchy-Goursat Theorem) *Let f be analytic in a simply connected domain D. If C is any simple closed contour in D, then*

$$\int_C f(z)\,dz = 0.$$

Example 4.7 $f(z) = z^2, \exp(z), \cos(z)$ *etc are entire functions so integral about any loop is zero.*

Theorem 4.4 *Let C_1 and C_2 be two simple closed positively oriented contours such that C_2 lies entirely in the interior of C_1. If f is an analytic function in a domain D that contains C_1 and C_2 both and the region between them, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Example 4.8 $f(z) = 1/z$. *Find $\int_C f(z)\,dz$ if C is any contour containing origin. Choose a circular contour inside C.*

Example 4.9 $\int_C \frac{1}{z-z_0}dz = 2\pi i$ *if C contains z_0.*

Example 4.10 *Find $\int_C \frac{2zdz}{z^2+2}$ where $C : |z| = 2$. Extend the Cauchy Goursat theorem to multiply connected domains.*

Antiderivative

Theorem 4.5 (Fundamental Theorem of Integration) *Let f be de• ned in a simply connected domain D and is analytic in D. If z_0 and z are points in D and C is any contour in D joining z_0 and z, then the function*

$$F(z) = \int_C f(z)\,dz$$

is analytic in D and $F'(z) = f(z)$.

De• nition 4.4 *If f is analytic in D and z_1 and z_2 are two points in D then the de• nite integral is de• ned as*

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

where F is an antiderivative of f.

Example 4.11 $\int_0^{2+i} z^2 dz = z^3/3\big|_0^{2+i} = \frac{2}{3} + i\frac{11}{3}$.

Example 4.12 $\int_1^{i} \cos z = \sin z|_1^i = \sin i - \sin 1.$

Example 4.13 $\int_{z_1}^{z_2} \frac{dz}{z} = \log z_2 - \log z_1$.

Cauchy Integral Formula

Theorem 4.6 (Cauchy Integral Formula) *Let f be analytic in domain D. Let C be a positively oriented simple closed contour in D. If z_0 is in the interior of C then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0}.$$

Example 4.14 $f(z) = \frac{1}{z^2+4}$. *Find $\int_C f(z)\,dz$ if $C : |z - i| = 2$.*

Example 4.15 $f(z) = \frac{z}{2z+1}$. *Find $\int_C f(z)\,dz$ if C is square with vertices on $(\pm 2, \pm 2)$.*

Theorem 4.7 *If f is analytic at a point, then all its derivatives exist and are analytic at that point.*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

5 Series

Convergence of Sequences and Series

Example 5.1 $z_n = 1/n$

Example 5.2 $z_n = a^n$

Example 5.3 $z_n = -2 + i\,(-1)^n\,/n^2$

Definition 5.1 *An infinite sequece* $z_1, z_2, \ldots, z_n, \ldots$ *of complex numbers has a* limit z *if for each positive* ϵ, *there exists positive integer* N *such that*

$$|z_n - z| < \epsilon \qquad \text{whenever} \qquad n > N.$$

The sequences have only one limit. A sequence said to converge to z if z is its limit. A sequence diverges if it does not converge.

Example 5.4 $z_n = z^n$ *converges to 0 if* $|z| < 1$ *else diverges.*

Example 5.5 $z_n = 1/\sqrt{n} + i\,(n+1)\,/n$ *converges to* i.

Theorem 5.1 *Suppose that* $z_n = x_n + iy_n$ *and* $z = x + iy$. *Then,*

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \lim_{n\to\infty} y_n = y$$

Definition 5.2 *If* $\{z_n\}$ *is a sequence, the infinite sum* $z_1 + z_2 + \cdots + z_n + \cdots$ *is called a series and is denoted by* $\sum\limits_{n=1}^{\infty} z_n$.

Definition 5.3 *A series* $\sum\limits_{n=1}^{\infty} z_n$ *is said to converge to a sum* S *if a sequence of partial sums*

$$S_N = z_1 + z_2 + \cdots + z_n$$

converges to S.

Theorem 5.2 *Suppose that* $z_n = x_n + iy_n$ *and* $S = X + iY$*. Then,*

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \qquad \text{and} \qquad \sum_{n=1}^{\infty} y_n = Y$$

Example 5.6 $\sum_0^\infty z^n = 1/(1-z)$ *if* $|z| < 1$

Taylor Series

Theorem 5.3 (Taylor Series) *If* f *is analytic in a circular disc of radius* R_0 *andcentered at* z_0*, then at each point inside the disc there is a series representation for* f *given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Example 5.7 $\sin z = \sum_0^\infty z^{2n+1}/(2n+1)!.$

Example 5.8 $\frac{1}{1+z} = \sum_0^\infty (-1)^n z^n \qquad |z| < 1.$

Example 5.9 $e^z = \sum_0^\infty z^n/n!.$

Example 5.10 $\frac{1}{z} = \sum_0^\infty (-1)^n (z-1)^n \qquad |z-1| < 1.$

Laurent Series

Theorem 5.4 (Laurent Series) *If* f *is analytic at all points* z *in an annular region* D *such that* $R_1 < |z - z_0| < R_2$*, then at each point in* D *there is a series representation for* f *given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \end{aligned}$$

and C is any contour in D.

Example 5.11 *If f is analytic inside a disc of radius R about z_0, then the Laurent series for f is identical to the Taylor series for f. That is all $b_n = 0$.*

Example 5.12 $\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n}$ *where* $|z| > 1$.

Example 5.13 $f(z) = \frac{-1}{(z-1)(z-2)}$. *Find Laurent series for all* $|z| < 1$, $1 < |z| < 2$ *and* $|z| > 2$.

Example 5.14 *Note that* $b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$.

6 Theory of Residues And Its Applications

Singularities

De• nition 6.1 *If a function f fails to be analytic at z_0 but is analytic at some point in each neighbourhood of z_0, then z_0 is a* $\mathrm{singular\ point}$ *of f.*

De• nition 6.2 *If a function f fails to be analyitc at z_0 but is analytic at each z in $0 < |z - z_0| < \delta$ for some δ, then f is said to be an* $\mathrm{isolated\ singular\ point}$ *of f.*

Example 6.1 $f(z) = 1/z$ *has an isolated singularity at* 0.

Example 6.2 $f(z) = 1/\sin(\pi z)$ *has isolated singularities at* $z = 0, \pm 1, \ldots$.

Example 6.3 $f(z) = 1/\sin(\pi/z)$ *has isolated singularities at* $z = 1/n$ *for integral n, also has a singularity at* $z = 0$.

Example 6.4 $f(z) = \log z$ *all points of negative x-axis are singular.*

Types of singularities

If a function f has an isolated singularity at z_0 then $\exists$ a δ such that f is analytic at all points in $0 < |z - z_0| < \delta$. Then f must have a Laurent series expansion about z_0. The part $\sum_{n=1}^{\infty} b_1 (z - z_0)^{-n}$ is called the principal part of f.

1. If there are in• nite nonzero b_i in the prinipal part then z_0 is called an essential singularity of f.
2. If for some integer m, $b_m \neq 0$ but $b_i = 0$ for all $i > m$ then z_0 is called a pole of order m of f. If $m = 1$ then it is called a simple pole.
3. If all bs are zero then z_0 is called a removable singularity.

Example 6.5 *$f(z) = (\sin z)/z$ is unde•ned at $z = 0$. f has a removable isolated singularity at $z = 0$.*

Residues

Suppose a function f has an isolated singularity at z_0, then there exists a $\delta > 0$ such that f is analytic for all z in deleted nbd $0 < |z - z_0| < \delta$. Then f has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

The coef• cient

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

where C is any contour in the deleted nbd, is called the residue of f at z_0.

Example 6.6 *$f(z) = \frac{1}{z-z_0}$. Then $\int_C f(z)\, dz = 2\pi i b_1 = 2\pi i$ if C contains z_0, otherwise 0.*

Example 6.7 *$f(z) = \frac{1}{z(z-z_0)^4}$. Show $\int_C f(z)\, dz = -\frac{\pi i}{8}$ if $C : |z - 2| = 1$.*

Example 6.8 *$f(z) = z \exp\left(\frac{1}{z}\right)$. Show $\int_C f(z)\, dz = \pi i$ if $C : |z| = 1$.*

Example 6.9 *$f(z) = \exp\left(\frac{1}{z^2}\right)$. Show $\int_C f(z)\, dz = 0$ if $C : |z| = 1$ even though it has a singularity at $z = 0$.*

Theorem 6.1 *If a function f is analytic on and inside a positively oriented countour C, except for a • nite number of points $z_1, z_2, \ldots, z_k$ inside C, then*

$$\int_C f(z)\, dz = 2\pi i \sum_{i=1}^{k} \mathit{Res} f(z_i).$$

Example 6.10 *Show that $\int \frac{5z-2}{z(z-1)} dz = 10\pi i$*

Residues of Poles

Theorem 6.2 *If a function f has a pole of order m at z_0 then*

$$\mathit{Res} f(z_0) = \lim_{z \to z_0} \left[\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right) \right].$$

Example 6.11 *$f(z) = \frac{1}{z-z_0}$. Simple pole Res $f(z_0) = 1$.*

Example 6.12 $f(z) = \frac{1}{z(z-z_0)^4}$. *Simple pole at* $z = 0$. *Res* $f(0) = 1/16$. *Pole of order* 4 *at* $z = 2$. *Res* $f(2) = -1/16$.

Example 6.13 $f(z) = \frac{\cos z}{z^2(z-\pi)^3}$. *Res* $f(0) = -3/\pi^4$. *Res* $f(\pi) = -\left(6 - \pi^2\right)/2\pi^4$.

Quotients of Analytic Functions

Theorem 6.3 *If a function* $f(z) = \frac{P(z)}{Q(z)}$, *where* P *and* Q *are analytic at* z_0, *then*

1. f is singular at z_0 iff $Q(z_0) = 0$.
2. f has a simple pole at z_0 if $Q'(z_0) \neq 0$. Then residue of f at z_0 is $P(z_0)/Q'(z_0)$.

A References

This appendix contains the references.

B Index

This appendix contains the index.