# Chapter 2

# Fourier Series and Transforms

## 2.1 Fourier Series

Let $f(x)$ be an integrable functin on $[-L, L]$. Then the fourier co-efficients are defined as

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n \geq 0, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n > 0. \end{aligned} \tag{2.1}$$

The claim is that the function $f$ then can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right].$$

The rhs is called the fourier series of $f$. Using fourier trick, one can obtain the expressions for $a_n$ and $b_n$. Multiplying both sides of the expression by $\cos\left(\frac{m\pi x}{L}\right)$ for some $m > 0$ and integrating one gets

$$\begin{aligned} & \int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx \\ = & \int_{-L}^{L}\frac{a_0}{2}\cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx + b_n\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx\right] \\ = & \ 0 + \sum_{n=1}^{\infty}\left[a_n L\delta_{m,n} + 0\right] = La_m \end{aligned}$$

This proves the formula given by Equation 2.1.

**Example 22.** Let $f(x) = x$ on $[-\pi, \pi]$. Find fourier series.

The function is odd, hence for all $n$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = 0$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \frac{2}{n}(-1)^{n+1}$$

Thus the fourier series is

$$f(x) = 2\left[\sin(x) - \frac{1}{2}\sin(2x) + \cdots\right]$$

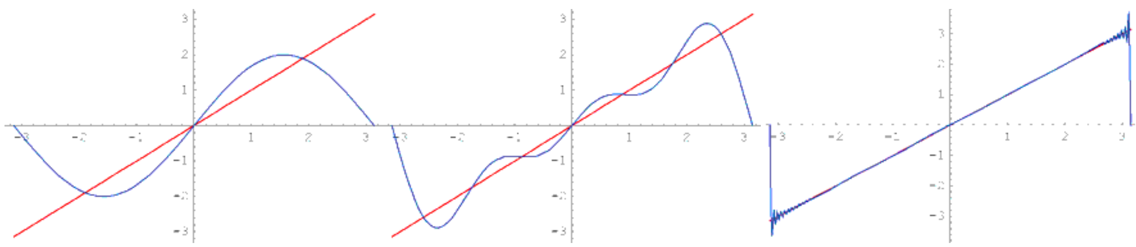


Figure 2.1: Fourier series with 1 term, 3 terms and 100 terms

Two points can be noted from the example

1. The convergence is not uniform, some points converge faster than other points.
2. At $x = \pm\pi$ the fourier series does not converge to $f$.

**Example 23.** Let $f(x) = 1 - |x|/L$ where $x \in [-L, L]$. This is a trianglular function. The function is even, thus fourier series will contain only cos terms.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0.$$

and for $n > 0$,

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\int_{-L}^{0}\left(1+\frac{x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L}\left(1-\frac{x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{n^2\pi^2}\left(1-(-1)^n\right) \\
&= \frac{4}{n^2\pi^2} \quad \text{odd } n \\
&= 0 \quad \text{even } n
\end{aligned}$$

Simillarly, $a_0 = 1$. The fourier series is

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\cos\left(\frac{\pi x}{L}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{L}\right) + \cdots\right].$$

## Dirichlet Condition

**Definition 24.** A function $f$ on $[-L, L]$ is said to be satisfy Dirichlet conditions if

- ▷ $f$ has finite number of extrema,
- ▷ $f$ has finite number of discontinuities,
- ▷ $f$ is abosultely integrable,
- ▷ $f$ is bounded.

**Theorem 25.** *Suppose a function $f$ satisfies Dirichlet conditions. Then the fourier series of $f$ converges to $f$ at points where $f$ is continuous. The fourier series converges to the midpoint of discontinuity at points where $f$ is discontinuous.*

**Example 26.** A square function $f$ is defined as

$$f(x) = \begin{cases} 0 & x \in [-L, 0] \\ 1 & x \in [0, L] \end{cases}$$

The fourier coefficients are

$$\begin{aligned} a_0 &= 1, \quad a_n = 0 \\ b_n &= \begin{cases} \frac{2}{n\pi} & \text{odd } n \\ 0 & \text{even } n \end{cases} \end{aligned}$$

The fourier series is

$$\frac{1}{2} + \frac{2}{\pi}\sin\left(\frac{\pi x}{L}\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \cdots$$

Clearly, at $x = 0$ series has a value 0.5 which is equal to $[f(0+) + f(0-)]/2$.

**Theorem 27.** *If a function $f$ on $[-L, L]$ is square integrable, that is*

$$\int_{-L}^{L} |f(x)|^2\, dx$$

*exists, then the fourier series of $f$ converges to $f$ almost everywhere.*

### Exponential Fourier Series

Substituting $\cos x = [e^x + e^{-x}]/2$ and $\sin x = [e^x - e^{-x}]/2i$, the fourier series of a function $f$ can be rewritten as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \\ &= \sum_{n=-\infty}^{\infty} c_n \exp\left( i \frac{n\pi x}{L} \right) \end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_n + ib_n) & n < 0 \\ \frac{a_0}{2} & n = 0 \end{cases}$$

Or,

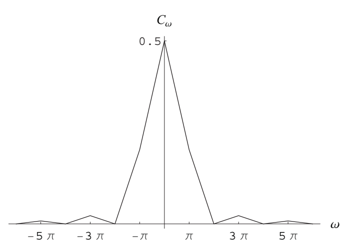
$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \exp\left( -i \frac{n\pi x}{L} \right)$$

Typically, if the argument of $f$ is a time variable, then $\omega_n = n\pi/L$ are called frequencies. The fourier coefficients are usually labelled by frequencies, that is $c_n$ is written as $c_{\omega_n}$.

**Example 28.** Exponential fourier coefficients for triangular function are given by

$$\begin{aligned} c_0 &= \frac{1}{2} \\ c_{2n} &= 0 \\ c_{2n-1} &= \frac{2}{(2n-1)^2\pi^2} \end{aligned}$$

The following graph shows the fourier coefficients as a function of frequencies.




## 2.2 Fourier Transform

For an interval $[-L, L]$, the fourier frequencies are given by $\omega_n = n\pi/L$. When $L \to \infty$, the fourier frequencies become a continuous variable. To get this idea, redefine the fourier coefficients

$$c_{\omega_n} = \int_{-L}^{L} f(x) \exp(-i\omega_n x)$$

Then the fourier series becomes

$$f(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} c_{\omega_n} \exp(i\omega_n x)$$

Now, let $\Delta\omega = \pi/L$, then

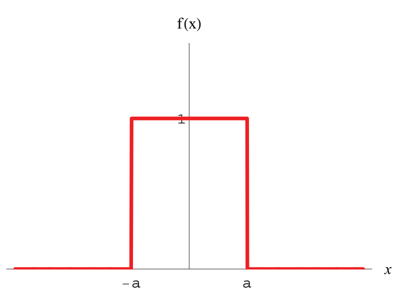
$$\begin{aligned} f(x) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_{\omega_n} \exp(i\omega_n x) \Delta\omega \\ &\to \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) \exp(i\omega x)\, d\omega \qquad \text{as } L \to \infty \end{aligned}$$

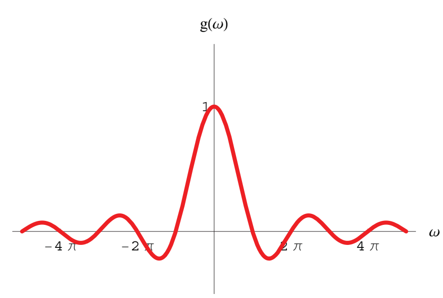
The last step is just the definition of Riemann integral. The function $c(\omega)$ is called as fourier transform of $f(x)$. It is denoted as $c = \mathcal{F}(f)$. Different text books define fourier transform differently by placing the constat factor $2\pi$ differently. In this note, fourier transform is defined as

$$\begin{aligned} c(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-i\omega x)\, dx \\ f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\omega) \exp(i\omega x)\, d\omega. \end{aligned}$$

**Example 29.** Let $f(x) = 1$ for $|x| \le a$, and is 0 otherwise. If $g$ is the FT of $f$, then

$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-i\omega x)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp(-i\omega x)\, dx \\ &= \frac{2a}{\sqrt{2\pi}} \frac{\sin(\omega a)}{\omega a} \end{aligned}$$




Here is another expample that illustrates the uncertainty principle.

**Example 30.** Let $f(x) = \sqrt{a/\pi}\exp[-ax^2]$. If $g$ is the FT of $f$, then

$$\begin{aligned}
g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp\left(-i\omega x\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} \exp\left(-ax^2 - i\omega x\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a}{\pi}} e^{-\omega^2/4a} \int_{-\infty}^{\infty} \exp\left(-a\left(x + \frac{i\omega}{2a}\right)^2\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{4a}\right)
\end{aligned}$$

The width of the gaussian function is defined by parameter $a$. Thus $f(x)$ is broad if $a$ is small. And $g(\omega)$ is narrow and sharp for small $a$.

Here is a very useful theorem called Parseval theorem.

**Theorem 31.** *If $g$ is the fourier transform of a square integrable function $f$ then*

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |g(\omega)|^2 \, d\omega.$$

## Uncertainty Relation

The uncertainty relation is built into the fourier transform. It may be interpreted differently depending on the situation in which it is applied. Here we state the theorem without proof.

**Theorem 32.** *Suppose $g$ is a fourier transform of a square integrable function $f$ that is normalized to unity, that is*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$$

*. Let*

$$\begin{aligned}
\mu_f &= \int_{-\infty}^{\infty} x|f(x)|^2 dx \\
\sigma_f^2 &= \int_{-\infty}^{\infty} (x-\mu_f)^2 |f(x)|^2 dx \\
\mu_g &= \int_{-\infty}^{\infty} \omega|f(\omega)|^2 d\omega \\
\sigma_g^2 &= \int_{-\infty}^{\infty} (\omega-\mu_\omega)^2 |f(\omega)|^2 d\omega
\end{aligned}$$

*then $\sigma_f \sigma_g \geq \frac{1}{2}$.*

The quantities $\sigma$ are called uncertainties.

**Example 33.** Let $f(x) = (a/\pi)^{1/4} \exp[-ax^2/2]$. Then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$$

Clearly, $\mu_f = 0$ and

$$\sigma_f^2 = \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx = \frac{1}{2a}$$

If $g$ is the FT of $f$, then

$$g(\omega) \quad = \quad \left(\frac{1}{\pi a}\right)^{1/4} \exp\left(-\frac{\omega^2}{2a}\right).$$

Note that

$$\int_{-\infty}^{\infty} |g(\omega)|^2 d\omega = 1$$

Now $\mu_g = 0$. Then

$$\sigma_g^2 = \int_{-\infty}^{\infty} \omega^2 |g(\omega)|^2 d\omega = \frac{a}{2}$$

Then,

$$\sigma_f \sigma_g = \frac{1}{2}$$