1. Prove the mean value theorem: Let P be an interior point of a volume$V$. Let y be a solution of the Laplace equation in V. Then $y(P)$ is the average of y over the surface of any sphere in V centered about P. [Hint: Use the integral equation.] Prove that the solution of the Laplace equation cannot have a maximum or a minimum in V.

2. Consider the Laplace equation $\nabla^2\phi = 0$ in a volume V with boundary S.

   (a) Prove using the Green's identity, that for a function f,

   $$\int_V \left(f\nabla^2 f + |\nabla f|^2\right) dv = \oint_S f\left(\nabla f \cdot \hat{\mathbf{n}}\right) dS.$$

   (b) Prove that the solution (assuming that it exists) to the Laplace equation in V with either Dirichlet or Neumann boundary conditions must be unique.

3. Prove that the Green's function for Laplace equation must be symmetric under exchange of its arguments, that is, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. [Note: This result is true for all self-adjoint operators. Try and prove this result for Green's functions of Sturm-Liouville equations.]

4. Show that the Dirichlet Green's function for the unbounded space between $z = 0$ and $z = L$ planes is given by

   $$G(\mathbf{r}, \mathbf{r}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right).$$

5. For the same geometry of problem 4, show that alternate form of the Green's function is

   $$G(\mathbf{r}, \mathbf{r}') = 2\sum_{m=-\infty}^{\infty}\int_0^k dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)}.$$

6. Consider two parallel conducting plates $z = 0$ and $z = L$. The potential on the $z = 0$ plate is zero, and on $z = L$ is given by

   $$\Phi(\rho, \phi, L) = \begin{cases} V & \rho \le a \\ 0 & \rho > a. \end{cases}$$

   (a) Show that the potential between the plates cane be written as

   $$\Phi(\rho, \phi, z) = V\int_0^{\infty} d\lambda J_1(\lambda) J_0(\lambda\rho/a)\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}$$