1. Starting with the generating function for the Bessel functions, show the following:

   (a) $J_n(x) = (-1)^n J_n(-x)$.

   (b) $\exp(iz\cos\theta) = \sum_{-\infty}^{\infty} i^m J_m(z)\, e^{im\theta}$.

   (c) $\cos x = J_0(x) + 2\sum_1^{\infty}(-1)^n J_{2n}(x)$ and $\sin x = 2\sum_0^{\infty}(-1)^n J_{2n+1}(x)$.

2. Starting with the series form for the Bessel functions, show the following:

   (a) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

   (b) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.

   (c) $x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2) J_n(x) = 0$.

3. Show that between any two consecutive zeroes of $J_n(x)$ there is one and only one zero of $J_{n+1}(x)$. To prove this, first prove that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \quad \text{and} \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n-1}(x)$$

   using problem 2(a) and 2(b).

4. Prove $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$using the integral representation of the Bessel functions.

5. A plane wave (wavelength λ) is incident normally on a circular apperture (radius a) as shown in the figure. Show that the amplitude of the wave emitted in the direction that makes an angle α with z axis is given by

$$\Phi \sim \int_0^a r dr \int_0^{2\pi} e^{ibr\cos\theta} d\theta$$

   where $b = 2\pi \sin\alpha/\lambda$. Thus, show that

$$\Phi \sim \frac{\lambda a}{\sin\alpha} J_1\left(\frac{2\pi a}{\lambda}\sin\alpha\right).$$

   Plot the intensity.

6. Show the following:

$$\begin{aligned}
\left(a^2 - b^2\right)\int_0^P J_n(ax)\, J_n(bx)\, x dx &= P\left[bJ_n(aP)\frac{d}{d(bx)}J_n(bP) - aJ_n(bP)\frac{d}{d(ax)}J_n(aP)\right], \\
\int_0^P \left[J_n(ax)\right]^2 x dx &= \frac{P^2}{2}\left\{\left[\frac{d}{d(ax)}J_n(aP)\right]^2 - \left(1 - \frac{n^2}{a^2P^2}\right)\left[J_n(aP)\right]^2\right\}, \\
\int_0^a \left[J_n\left(\rho_{nm}\frac{x}{a}\right)\right]^2 x dx &= \frac{a^2}{2}\left[J_{n+1}(\rho_{nm})\right]^2.
\end{aligned}$$

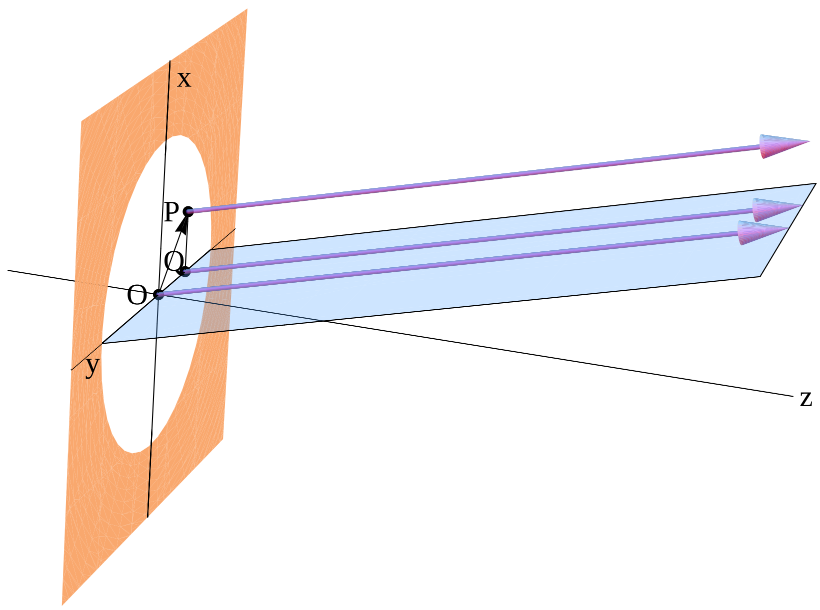

Figure 1: Problem 5: Coordinates of point P are $(r, \theta, z = 0)$ in cylindrical coordinates. All rays make an angle of α with the z axis.