1. For the operator $Ly(x) = y''$, find the Green's function with different boundary conditions given below:

   (a) $y(0) = y(1) = 0$,

   (b) $y(0) = 0$ and $y'(1) = 0$,

   (c) $y(-1) = y(1) = 0$.

2. Find the Green's Function for the following differential operators:

   (a) $Ly(x) = y''(x) + y(x)$, $x \in [0, 1]$, with $y(0) = 0$ and $y'(1) = 0$.

   (b) $Ly(x) = y''(x) - y(x)$, $x \in \mathbb{R}$, with $y(\pm\infty) < \infty$.

3. Find the Green's functions for the differential operators

$$Ly(x) = xy''(x) + y'$$

   with boundary conditions that $y(1) = 0$ and $y(0)$ should be finite. Use the Green's function, solve

$$\frac{d}{dx}\left[x\frac{dy}{dx}(x)\right] = -1.$$

   Verify the solution by direct integration of the differential equation.

4. Find the Green's function for the differential operator

$$\begin{aligned} Ly(x) &= xy''(x) + y'(x) - \frac{n^2}{x}y(x) \\ y(0) &< \infty \\ y(1) &= 0. \end{aligned}$$

5. Find the Green function for associated Legendre differential operator

$$Ly(x) = \frac{d}{dx}\left[\left(1 - x^2\right)\frac{dy}{dx}\right] - \frac{n^2}{(1 - x^2)}y \qquad x \in [-1, 1]$$

   with boundary condition that at ± 1, the solution must be finite.

6. Construct a Green's function to solve Helmholtz equation

$$y''(x) - k^2 y(x) = f(x)$$

   where, k is some constant. The boundary condition is that the Green's function must vanish as $x \to \pm\infty$.

7. From the eigenfunction expansion of the Green's function, show that

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\left(n + \frac{1}{2}\right)\pi x\right)\sin\left(\left(n + \frac{1}{2}\right)\pi t\right)}{\left(n + \frac{1}{2}\right)^2} = \begin{cases} x, & 0 \le x < t \\ t, & t < x \le 1. \end{cases}$$